Linear classifiers: Logistic regression

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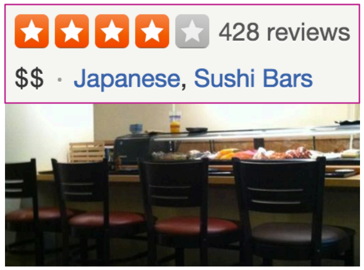
Predicting sentiment by topic: *An intelligent restaurant review system*

It’s a big day & I want to book a table at a nice Japanese restaurant



Positive reviews not positive about everything

Sample review:

Watching the chefs create incredible edible art made the experience very unique.

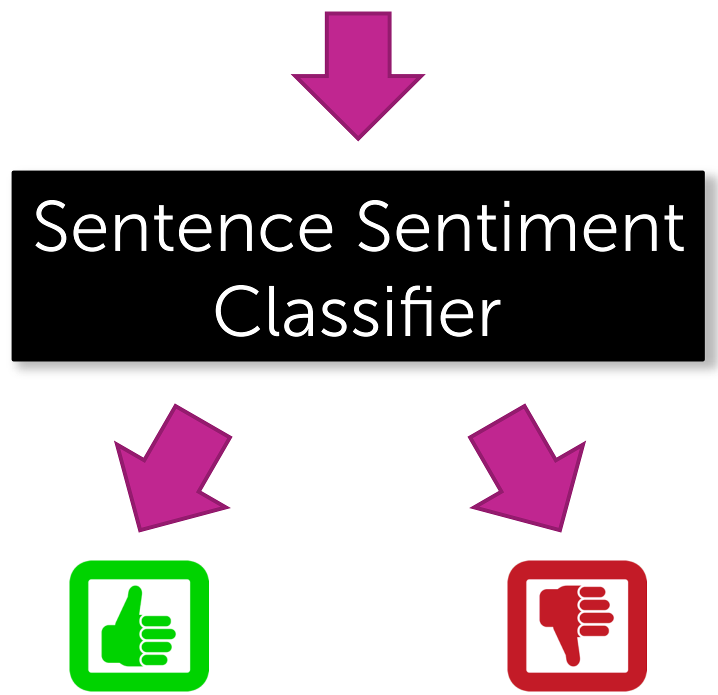
Experience

My wife tried their ramen and it was pretty forgettable.

All the sushi was delicious! Easily best sushi in Seattle.

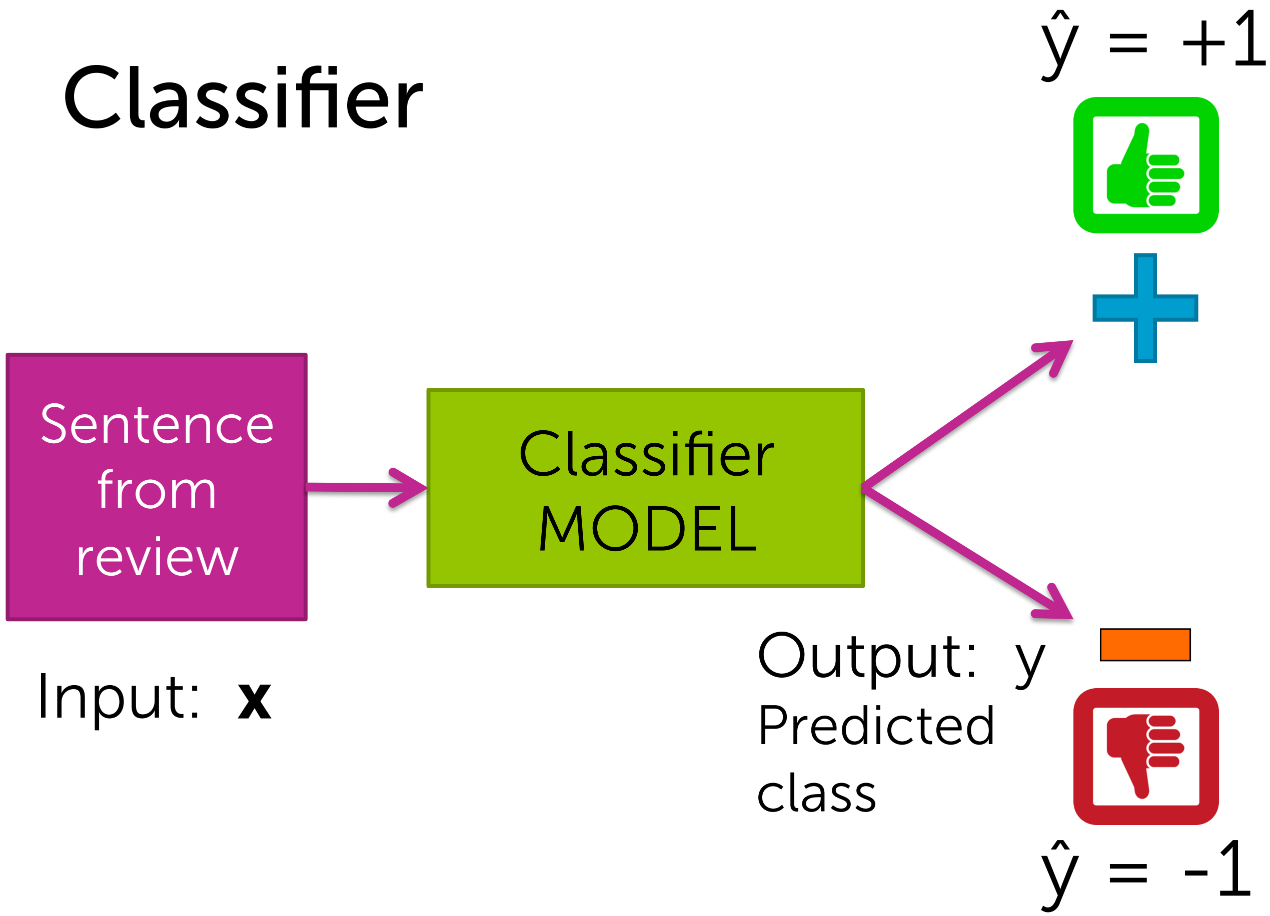
Classifying sentiment of review

Easily best sushi in Seattle.



Linear classifier: Intuition

Classifier



Note: we'll start talking about 2 classes, and address multiclass later

A (linear) classifier

- Will use training data to learn a weight or coefficient for each word

Word	Coefficient
good	1.0
great	1.5
awesome	2.7
bad	-1.0
terrible	-2.1
awful	-3.3
restaurant, the, we, where, …	0.0
…	…

Scoring a sentence

Word	Coefficient
good	1.0
great	1.2
awesome	1.7
bad	-1.0
terrible	-2.1
awful	-3.3
restaurant, the, we, where, …	0.0
…	…

Input $\mathbf{x}_i$:

Sushi was <u>great</u>,
the food was <u>awesome</u>,
but the service was <u>terrible</u>.

Called a linear classifier, because output is weighted sum of input.

Word	Coefficient
…	…

Simple linear classifier

Score(***x***) = weighted count of words in sentence

If *Score* (***x***) > *0*:

$\hat{y}$ = +1

Else:

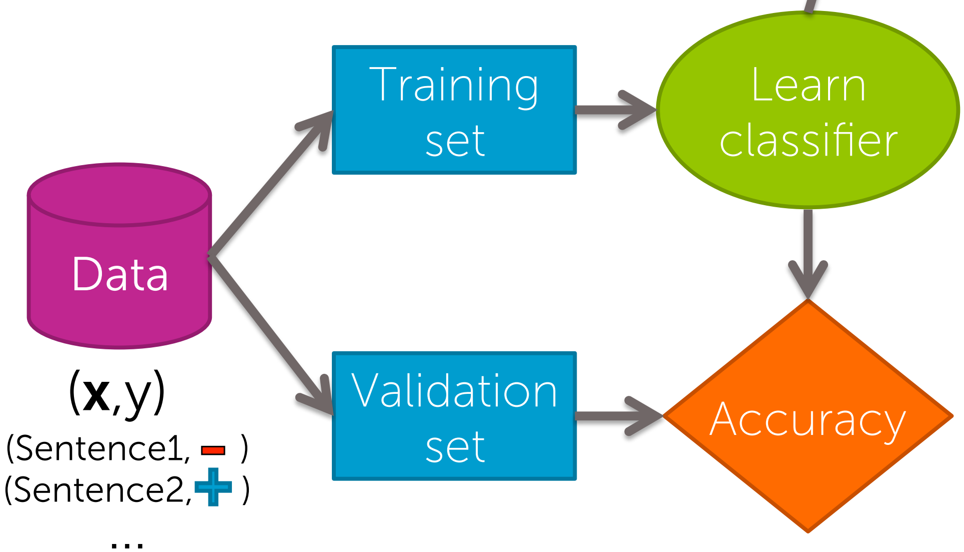
$\hat{y}$ = -1

Sentence from review

Input: **x**

Training a classifier = Learning the coefficients

Word	Coefficient
good	1.0
awesome	1.7
bad	-1.0
awful	-3.3
...	...

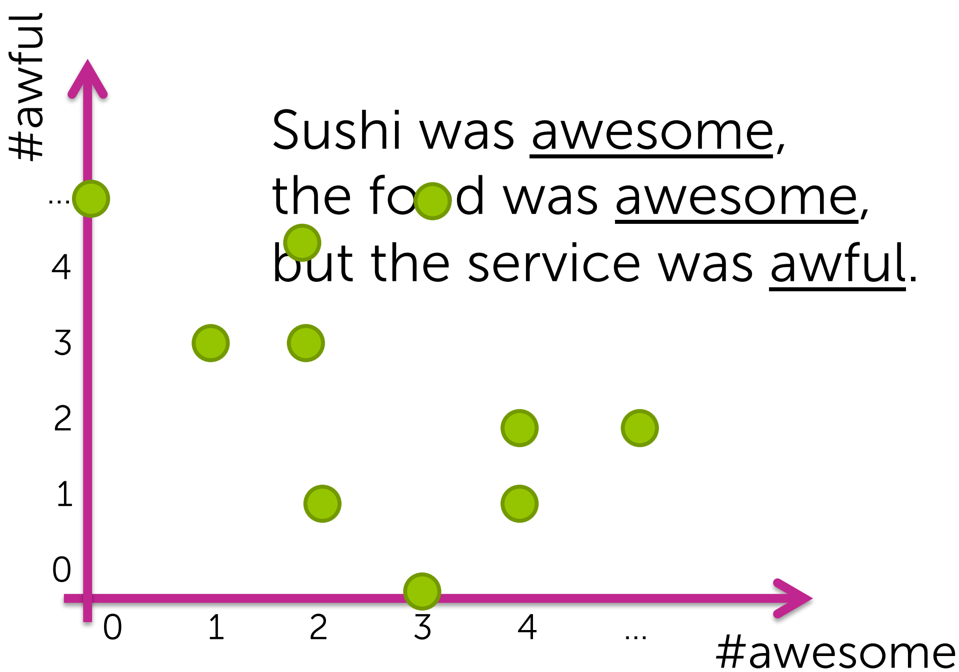


Decision boundaries

Suppose only two words had non-zero coefficient

Word	Coefficient
#awesome	1.0
#awful	-1.5

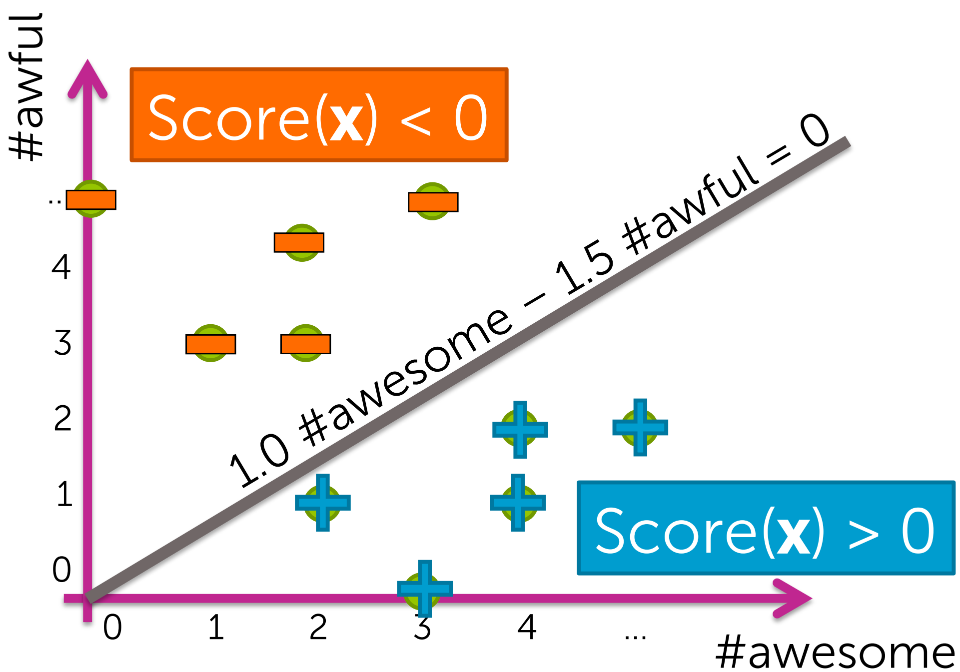
Score(x) = 1.0 #awesome – 1.5 #awful



Decision boundary example

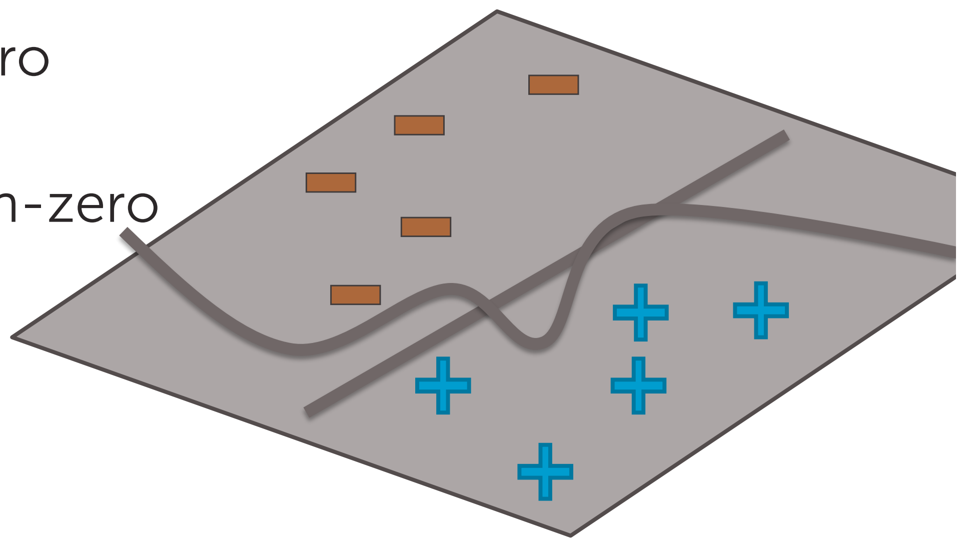
Word	Coefficient
#awesome	1.0
#awful	-1.5

Score(x) = 1.0 #awesome – 1.5 #awful

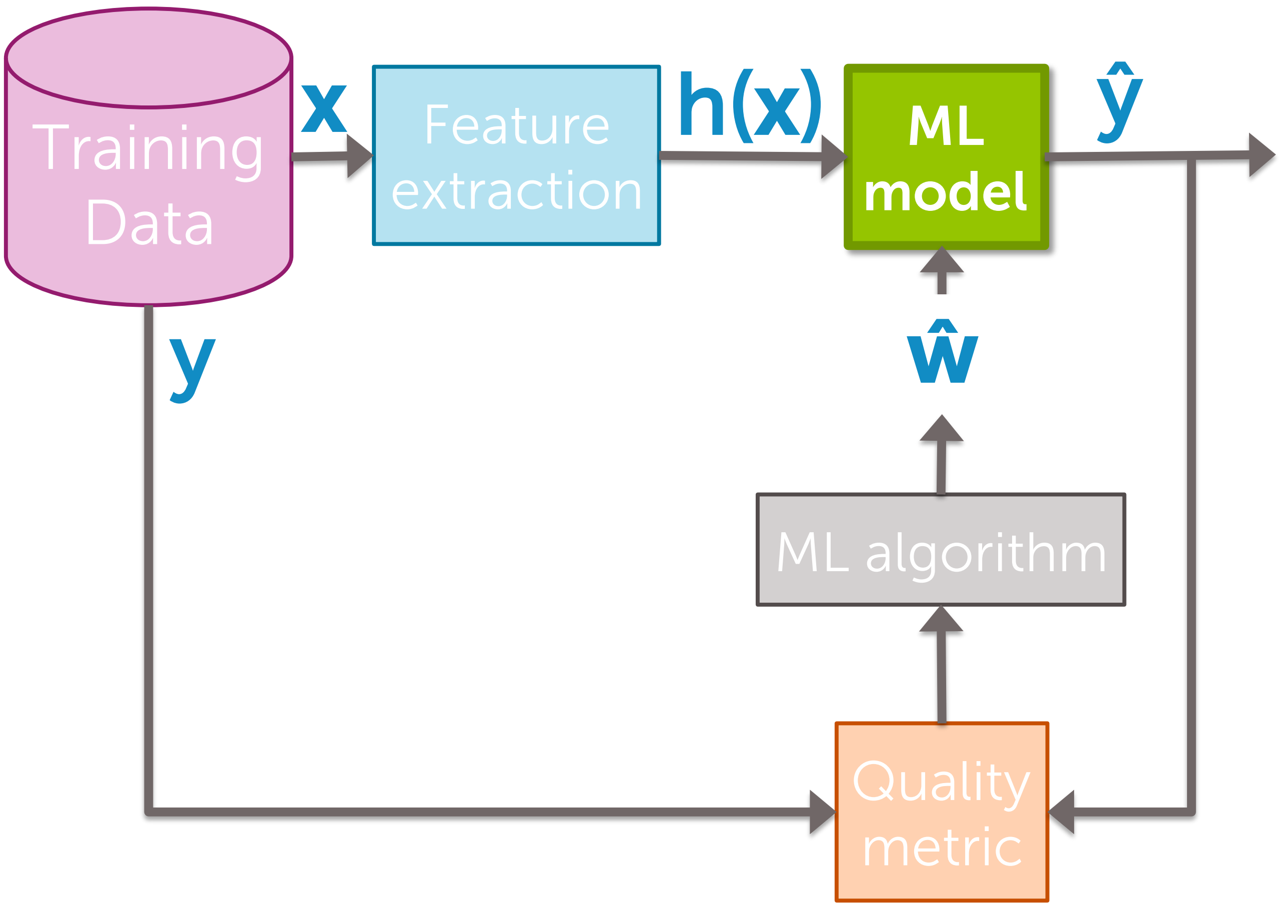


Decision boundary separates positive & negative predictions

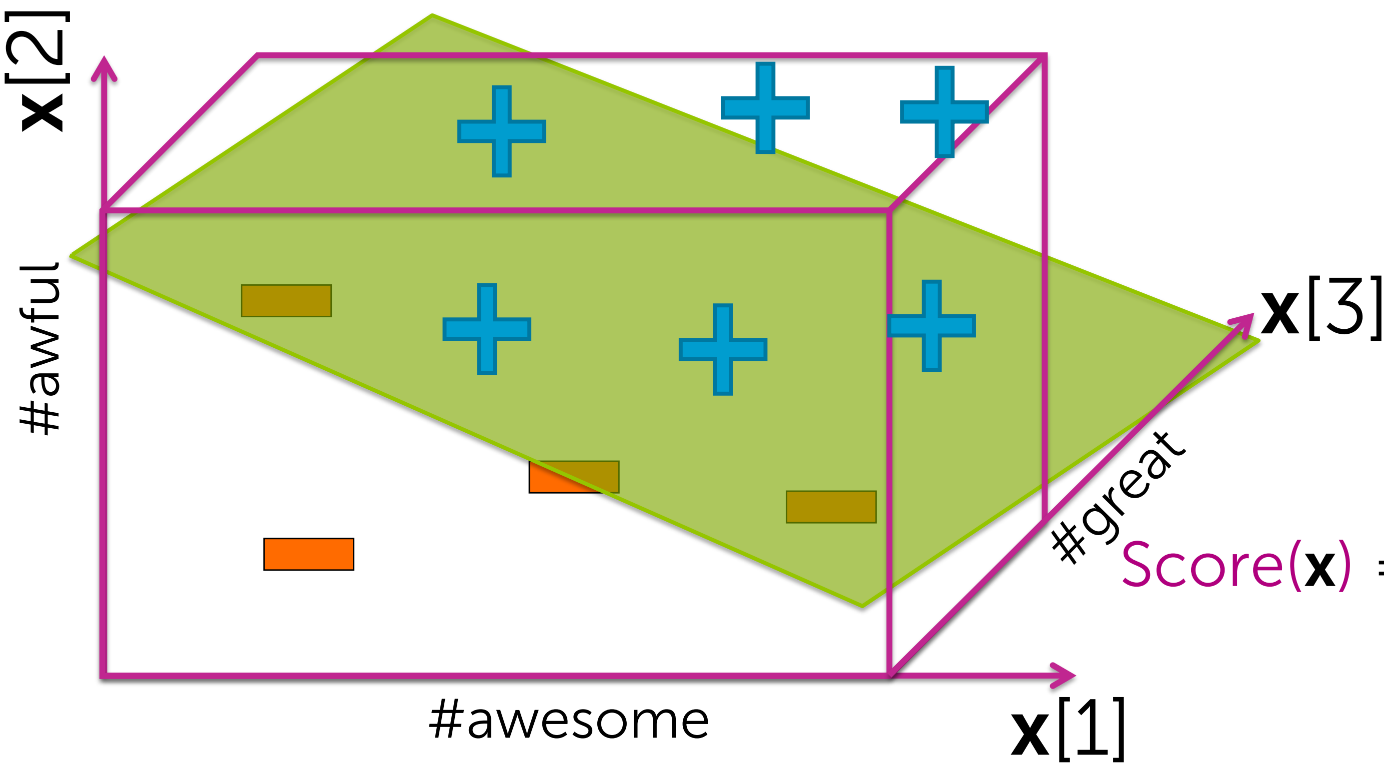
- For linear classifiers:
  - When 2 coefficients are non-zero
    ➔ **line**
  - When 3 coefficients are non-zero
    ➔ **plane**
  - When many coefficients are non-zero
    ➔ **hyperplane**
- For more general classifiers
  ➔ **more complicated shapes**



Linear classifier: Model


Training Data
x
Feature extraction
h(x)
ML model
ŷ
y
ŵ
ML algorithm
Quality metric

Coefficients of classifier



$$\text{Score}(\mathbf{x}) = w_0 + w_1 \text{ \#awesome} + w_2 \text{ \#awful} + w_3 \text{ \#great}$$

General notation

Output: y ← {-1,+1}

Inputs: $\mathbf{x}$ = ($\mathbf{x}$[1],$\mathbf{x}$[2],..., $\mathbf{x}$[d])

↖ d-dim vector

Notational conventions:

$\mathbf{x}$[j] = j^{th} input (*scalar*)

$h_j(\mathbf{x})$ = j^{th} feature (*scalar*)

$\mathbf{x}_i$ = input of i^{th} data point (*vector*)

$\mathbf{x}_i$[j] = j^{th} input of i^{th} data point (*scalar*)

Simple hyperplane

Model: $\hat{y}_i = \text{sign}(\text{Score}(\mathbf{x}_i))$

$\text{Score}(\mathbf{x}_i) = w_0 + w_1\, \mathbf{x}_i[1] + \ldots + w_d\, \mathbf{x}_i[d] = \mathbf{w}^T \mathbf{x}_i$

feature 1 = 1

feature 2 = $\mathbf{x}[1]$ … e.g., #awesome

feature 3 = $\mathbf{x}[2]$ … e.g., #awful

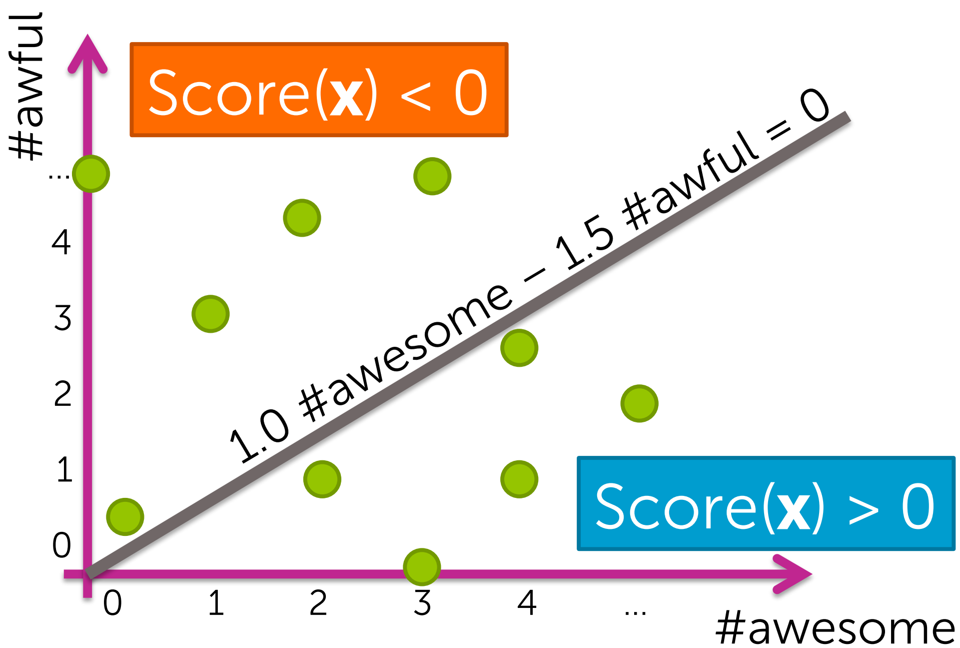
…

feature d+1 = $\mathbf{x}[d]$ … e.g., #ramen

Decision boundary: effect of changing coefficients

Input	Coefficient	Value
	w_0	0.0
#awesome	w_1	1.0
#awful	w_2	-1.5

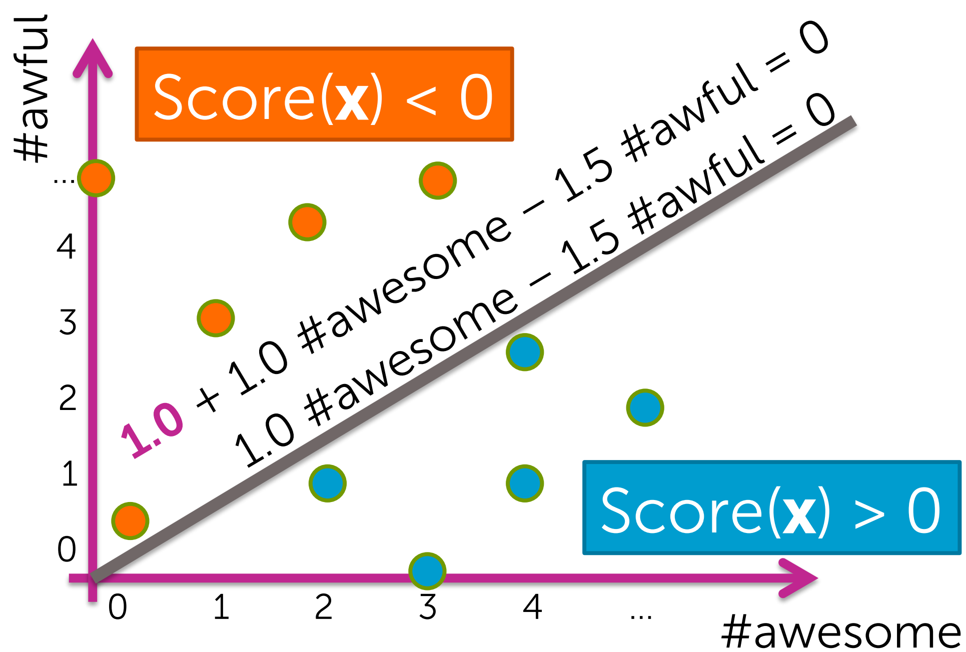
Score(x) = 1.0 #awesome – 1.5 #awful



Decision boundary: effect of changing coefficients

Input	Coefficient	Value
	w_0	**1.0**
#awesome	w_1	1.0
#awful	w_2	-1.5

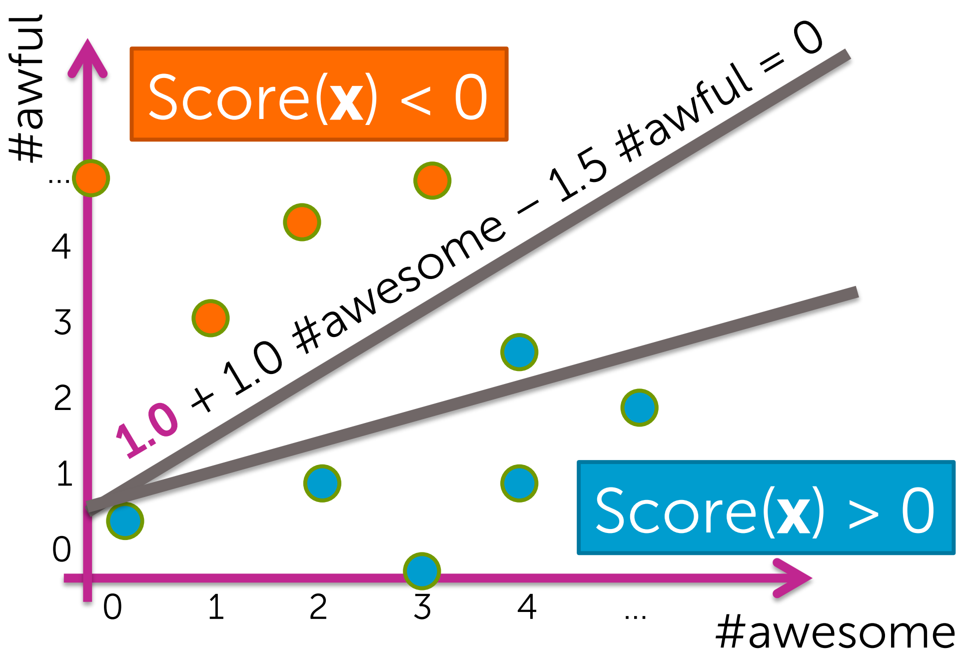
Score(x) = **1.0** #awesome – 1.5 #awful



Decision boundary: effect of changing coefficients

Input	Coefficient	Value
	w_0	**1.0**
#awesome	w_1	1.0
#awful	w_2	**–3.0**

Score(x) = **1.0** + 1.0 #awesome **– 3.0** #awful



More generic features… D-dimensional hyperplane

Model: $\hat{y}_i = \text{sign}(\text{Score}(\mathbf{x}_i))$

$$\text{Score}(\mathbf{x}_i) = w_0\, h_0(\mathbf{x}_i) + w_1\, h_1(\mathbf{x}_i) + \ldots + w_D\, h_D(\mathbf{x}_i)$$

$$= \sum_{j=0}^{D} w_j\, h_j(\mathbf{x}_i) = \mathbf{w}^T h(\mathbf{x}_i)$$

feature 1 = $h_0(\mathbf{x})$ … e.g., 1

feature 2 = $h_1(\mathbf{x})$ … e.g., $\mathbf{x}[1]$ = #awesome

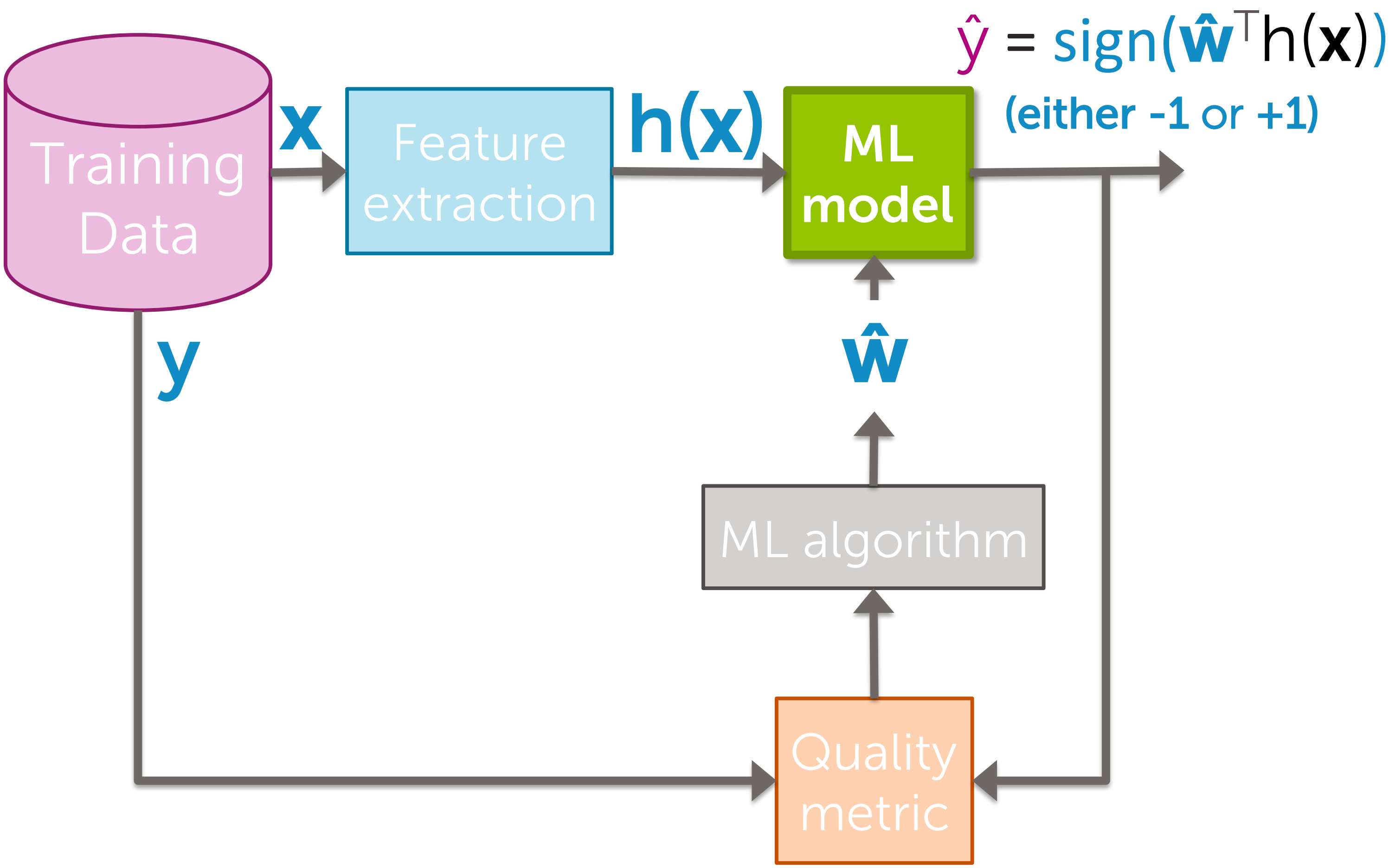
feature 3 = $h_2(\mathbf{x})$ … e.g., $\mathbf{x}[2]$ = #awful

or, $\log(\mathbf{x}[7])\ \mathbf{x}[2]$ = log(#bad) x #awful

or, tf-idf("awful")

…

feature D+1 = $h_D(\mathbf{x})$ … some other function of $\mathbf{x}[1]$,…, $\mathbf{x}[d]$



Training Data
x
Feature extraction
h(x)
ML model
ŷ = sign(ŵᵀh(x))
(either -1 or +1)
y
ŵ
ML algorithm
Quality metric

Are you sure about the prediction?
Class probability

How confident is your prediction?

- Thus far, we've outputted a prediction +1 or -1
- But, how sure are you about the prediction?

"The sushi & everything else were awesome!"

Definite +1

$\hat{y}$ = +1 with high probability

"The sushi was good, the service was OK"

Not sure

$\hat{y}$ = +1 with probability 0.5

Basics of probabilities – quick review

Basic probability

Probability a review is positive is 0.7

x = review text	y = sentiment
All the sushi was delicious! Easily best sushi in Seattle.	+1
The sushi & everything else were awesome!	+1
My wife tried their ramen, it was pretty forgettable.	-1
The sushi was good, the service was OK	+1
…	…

I expect 70% of rows to have y = +1
(Exact number will vary for each specific dataset)

Interpreting probabilities as degrees of belief

$P(y=+1)$

0.0 0.5 1.0

Absolutely sure reviews are negative

Absolutely sure reviews are positive

Not sure if reviews are positive or negative

Key properties of probabilities

Property	Two class (e.g., y is +1 or -1)	Multiple classes (e.g., y is dog, cat or bird)
Probabilities always between 0 & 1		
Probabilities sum up to 1		

Conditional probability

Probability a review with
3 "awesome" and 1 "awful" is positive is 0.9

x = review text	y = sentiment
All the sushi was delicious! Easily best sushi in Seattle.	+1
Sushi was **awesome** & everything else was **awesome**! The service was **awful**, but overall **awesome** place!	+1
My wife tried their ramen, it was pretty forgettable.	-1
The sushi was good, the service was OK	+1
…	…
awesome … **awesome** … **awful** … **awesome**	+1
…	…
awesome … **awesome** … **awful** … **awesome**	-1
…	…
…	…
awesome … **awesome** … **awful** … **awesome**	+1

I expect 90% of rows with reviews containing 3 "awesome" & 1 "awful" to have y = +1
(Exact number will vary for each specific dataset)

Interpreting conditional probabilities

Output label

Input sentence

$P(y=+1|\mathbf{x}_i=$ "All the sushi was delicious!"$)$

0.0 0.5 1.0

Absolutely sure review
"All the sushi was delicious!"
is negative

Not sure if review
"All the sushi was delicious!"
is positive or negative

Absolutely sure review
"All the sushi was delicious!"
is positive

Key properties of conditional probabilities

Property	Two class (e.g., y is +1 or -1, $\mathbf{x}_i$ is review text)	Multiple classes (e.g., y is dog, cat or bird, $\mathbf{x}_i$ is image)
Conditional probabilities always between 0 & 1		
Conditional probabilities sum up to 1 over y, but not over **x**		

Using probabilities in classification

How confident is your prediction?

"The sushi & everything else were awesome!"

Definite +1

P(y=+1|**x**=*"The sushi & everything else were awesome!"*)
= 0.99

"The sushi was good, the service was OK"

Not sure

P(y=+1|**x**=*"The sushi was good, the service was OK"*)
= 0.55

Many classifiers provide a degree of certainty:

Output label

Input sentence

P(y|**x**)

Extremely useful in practice

Goal: Learn conditional probabilities from data

Training data: N observations ($\mathbf{x}_i$,y_i)

$\mathbf{x}$[1] = #awesome	$\mathbf{x}$[2] = #awful	y = sentiment
2	1	+1
0	2	-1
3	3	-1
4	1	+1
…	…	…

Optimize **quality metric** on training data

Find best model $\hat{P}$ by finding best $\hat{\mathbf{w}}$

Useful for predicting $\hat{y}$

Sentence from review

Input: **x**

Predict most likely class

$\hat{P}(y|x)$ = estimate of class probabilities

If $\hat{P}(y=+1|x) > 0.5$:

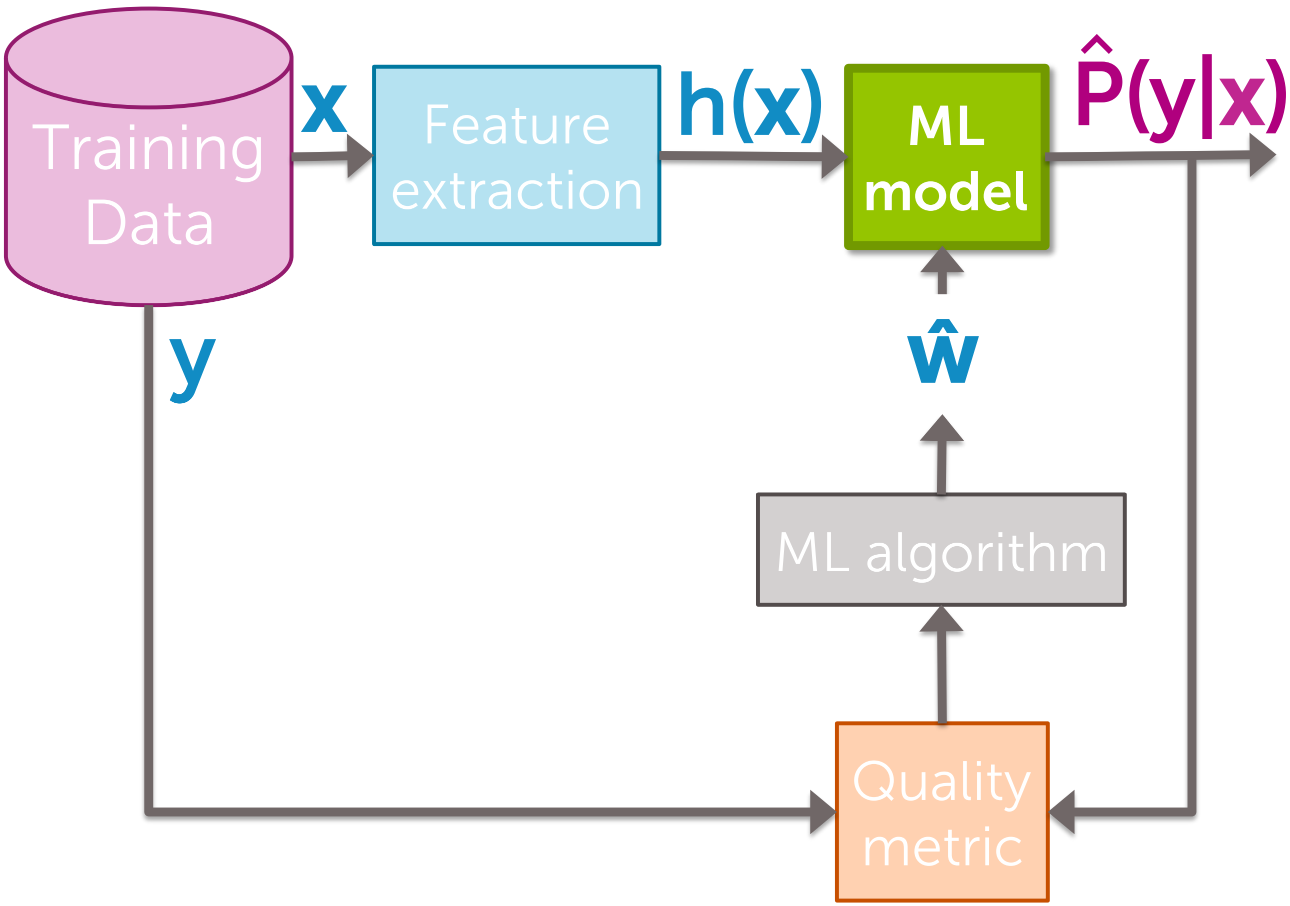
$\hat{y} = +1$

Else:

$\hat{y} = -1$

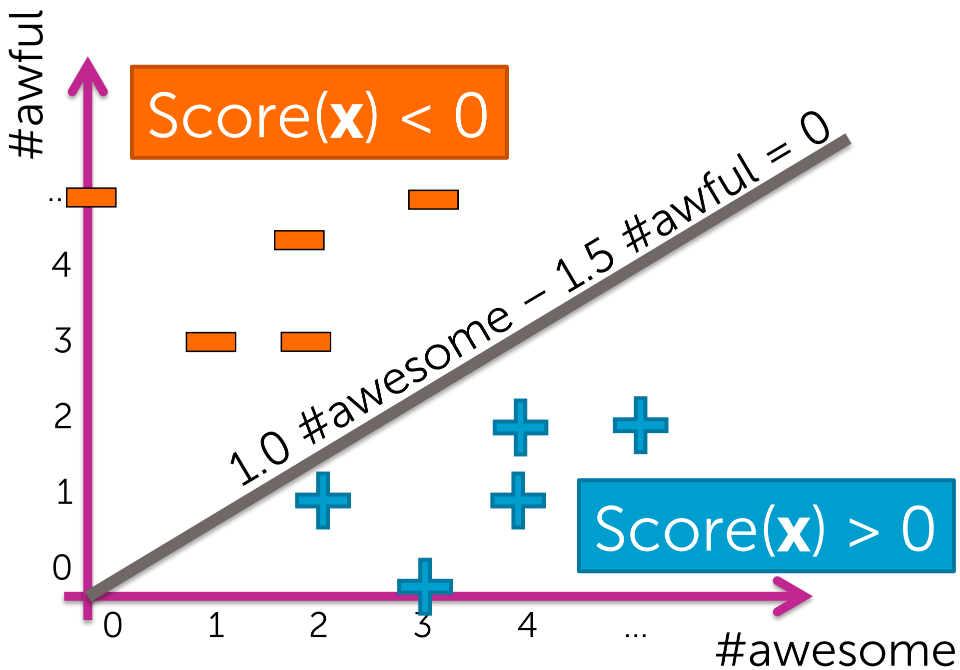
- Estimating $\hat{P}(y|x)$ improves **interpretability**:
  - Predict $\hat{y} = +1$ **and** tell me how sure you are

Predicting class probabilities with generalized linear models


Training Data
x
Feature extraction
h(x)
ML model
P̂(y|x)
y
ŵ
ML algorithm
Quality metric

Thus far, we focused on decision boundaries

$$\text{Score}(\mathbf{x}_i) = w_0\, h_0(\mathbf{x}_i) + w_1\, h_1(\mathbf{x}_i) + \ldots + w_D\, h_D(\mathbf{x}_i)$$
$$= \mathbf{w}^T h(\mathbf{x}_i)$$



Relate $\text{Score}(\mathbf{x}_i)$ to $\hat{P}(y=+1|\mathbf{x},\hat{\mathbf{w}})$?

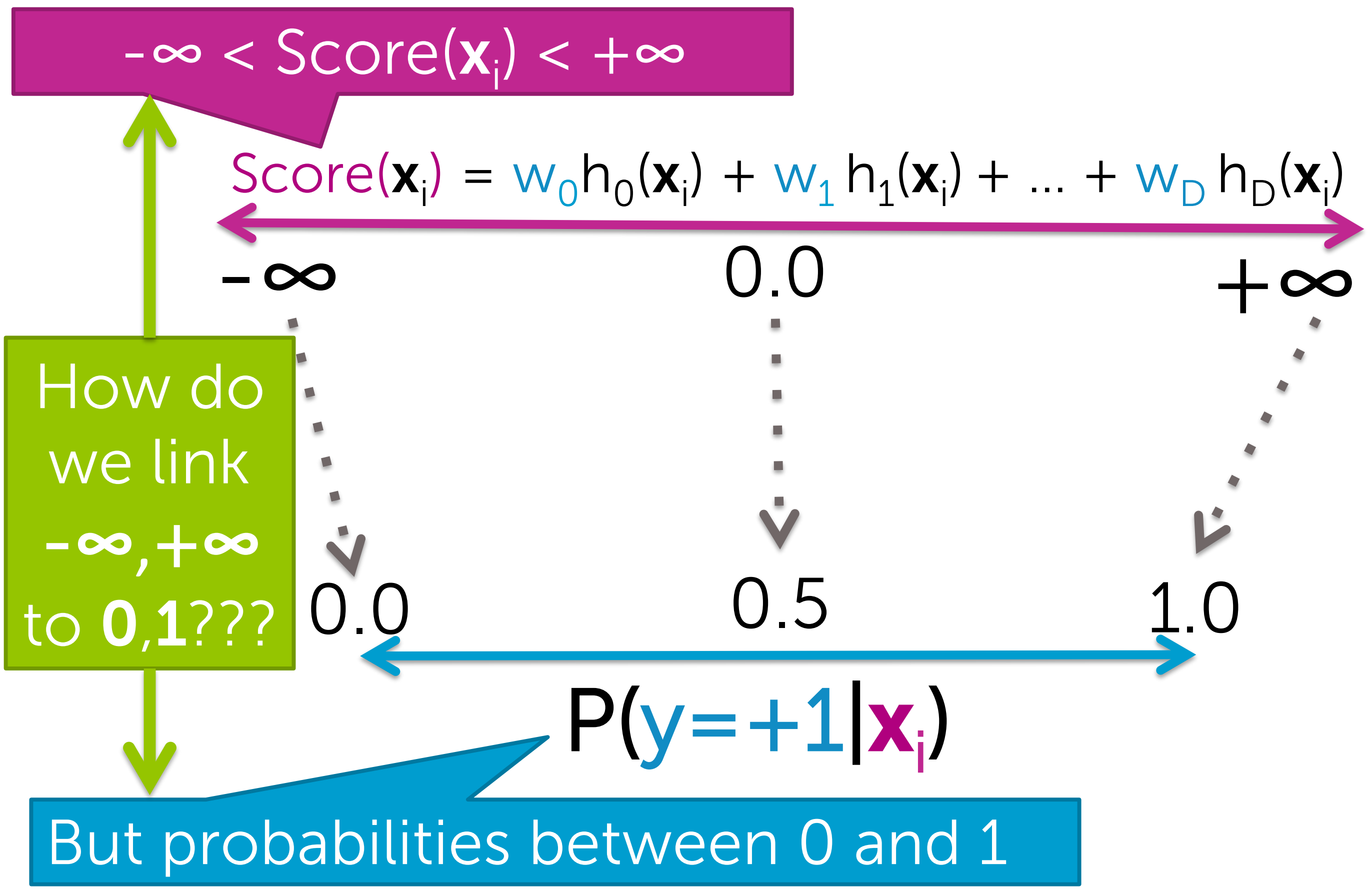
Interpreting Score($\mathbf{x}_i$)

$$\text{Score}(\mathbf{x}_i) = \mathbf{w}^T h(\mathbf{x}_i)$$

$-\infty$ $\qquad$ $\hat{y}_i = -1$ $\qquad$ 0.0 $\qquad$ $\hat{y}_i = +1$ $\qquad$ $+\infty$

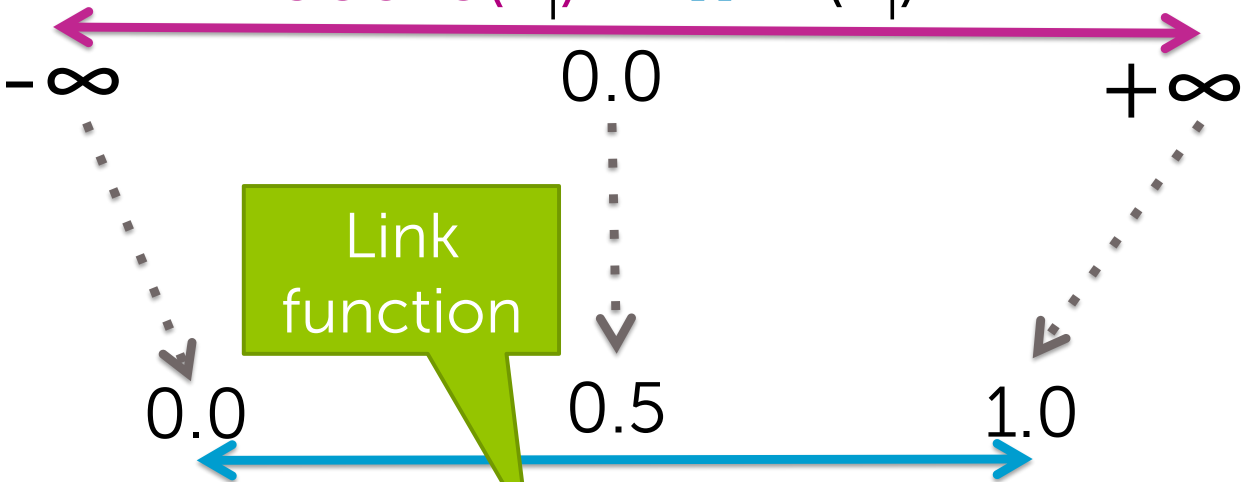
Very sure $\hat{y}_i = -1$	Not sure if $\hat{y}_i = -1$ or $+1$	Very sure $\hat{y}_i = +1$
$\hat{P}(y=+1\mid\mathbf{x}_i) = 0$	$\hat{P}(y=+1\mid\mathbf{x}_i) = 0.5$	$\hat{P}(y=+1\mid\mathbf{x}_i) = 1$

Why not just use regression to build classifier?



Link function: squeeze real line into [0,1]

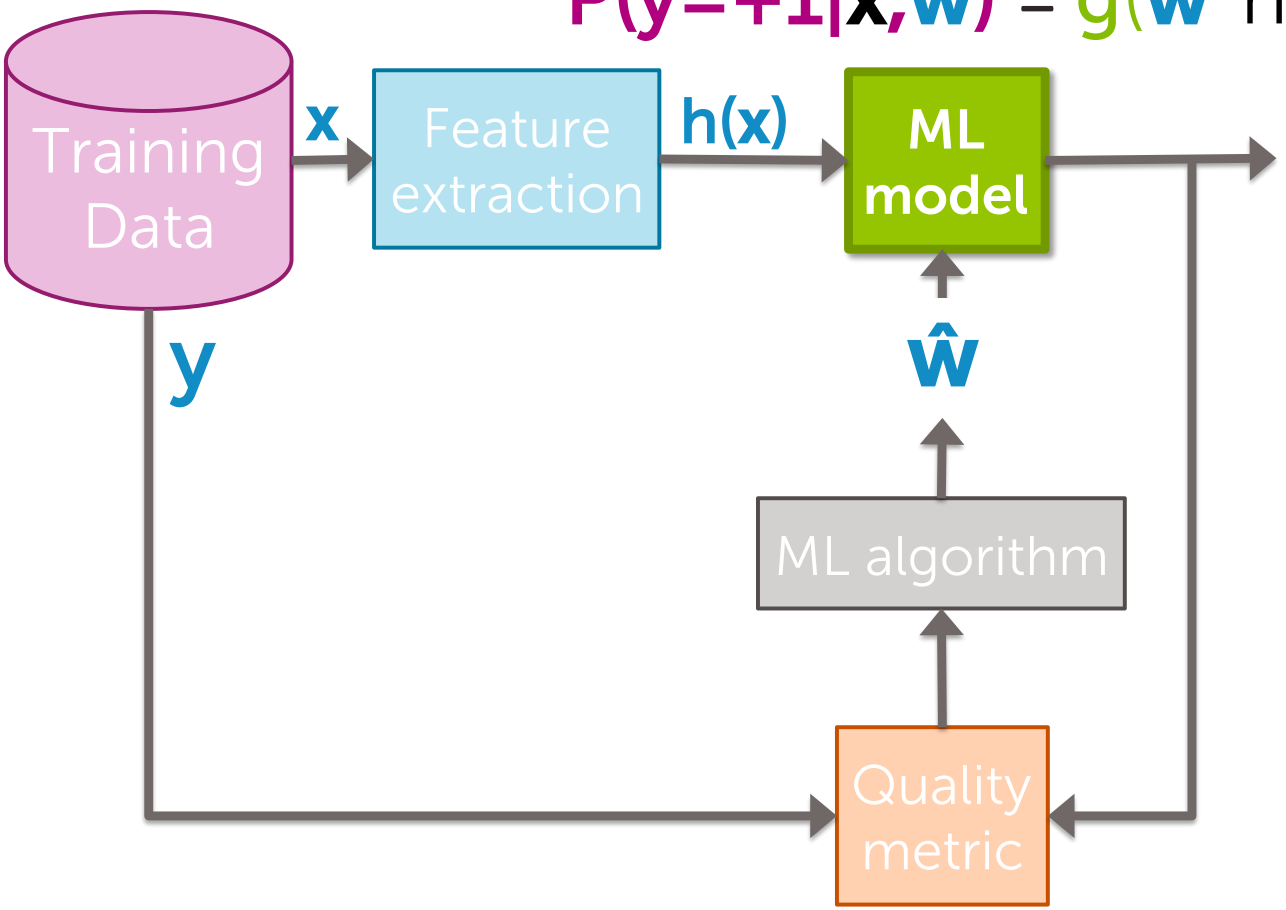
$\text{Score}(\mathbf{x}_i) = \mathbf{w}^T h(\mathbf{x}_i)$



$\hat{P}(y=+1|\mathbf{x}_i) = g(\mathbf{w}^T h(\mathbf{x}_i))$

Generalized linear model

$$\hat{P}(y=+1|\mathbf{x},\hat{\mathbf{w}}) = g(\hat{\mathbf{w}}^T h(\mathbf{x}))$$

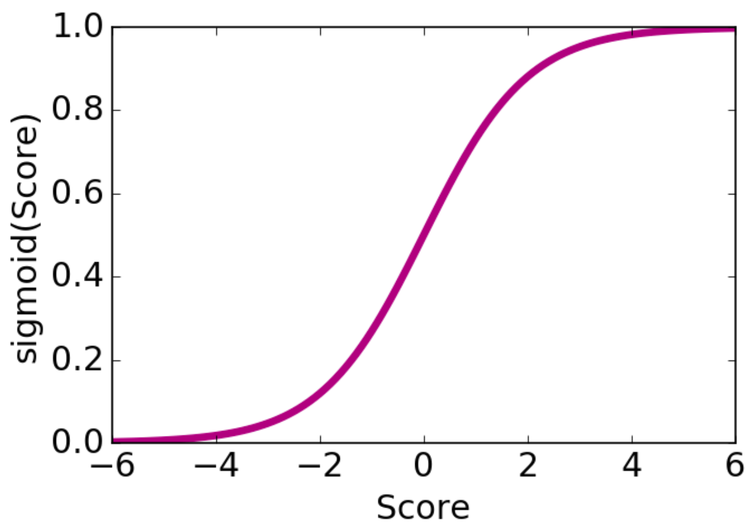


Logistic regression classifier:
linear score with logistic link function

Logistic function (sigmoid, logit)

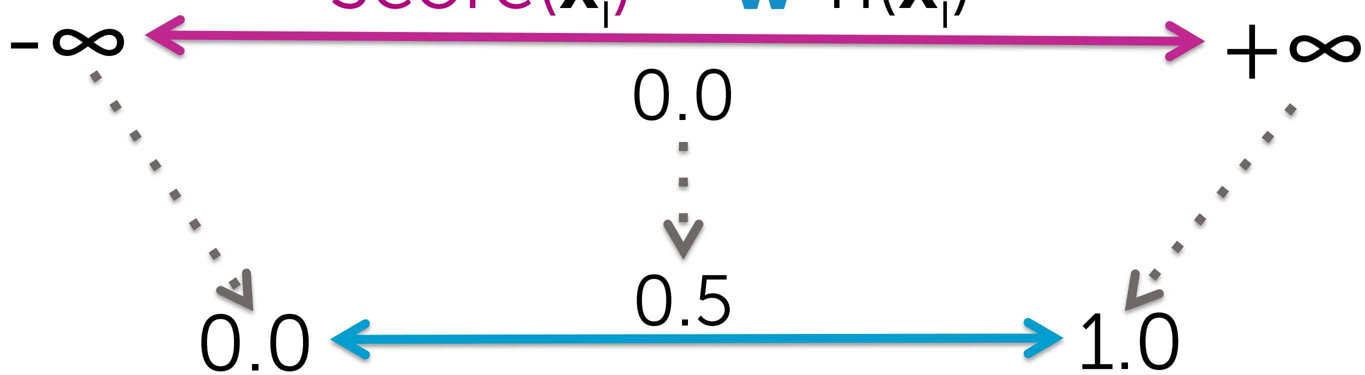
$$sigmoid(\text{Score}) = \frac{1}{1 + e^{-\text{Score}}}$$

Score	$-\infty$	-2	0.0	+2	$+\infty$
sigmoid(Score)					



Logistic regression model

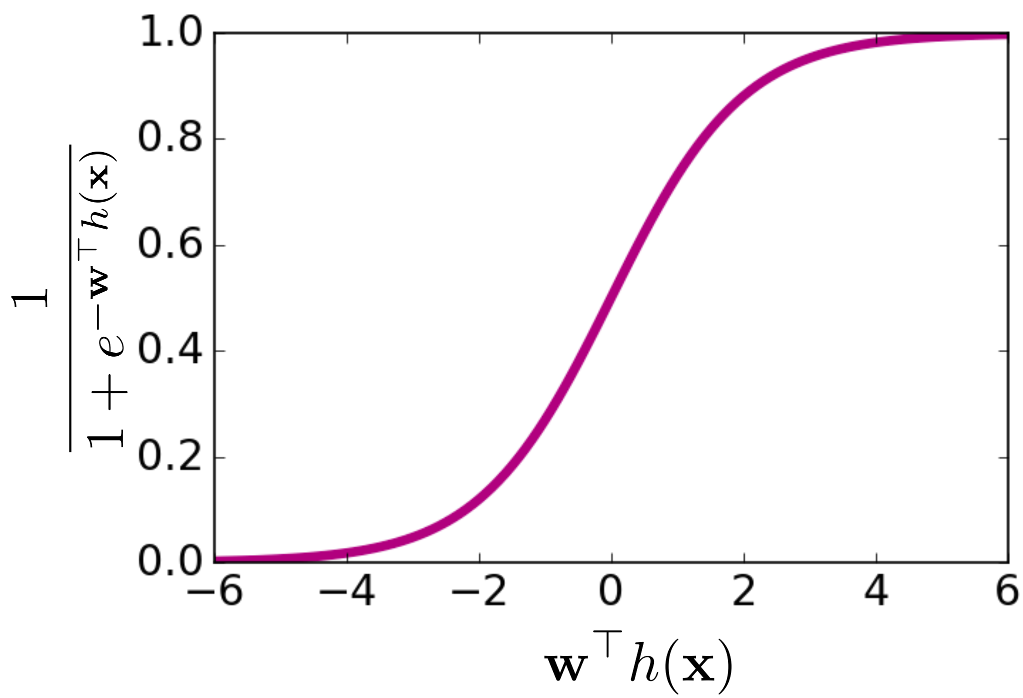
$$\text{Score}(\mathbf{x}_i) = \mathbf{w}^T h(\mathbf{x}_i)$$



$$P(y=+1|\mathbf{x}_i,\mathbf{w}) = \text{sigmoid}(\text{Score}(\mathbf{x}_i))$$

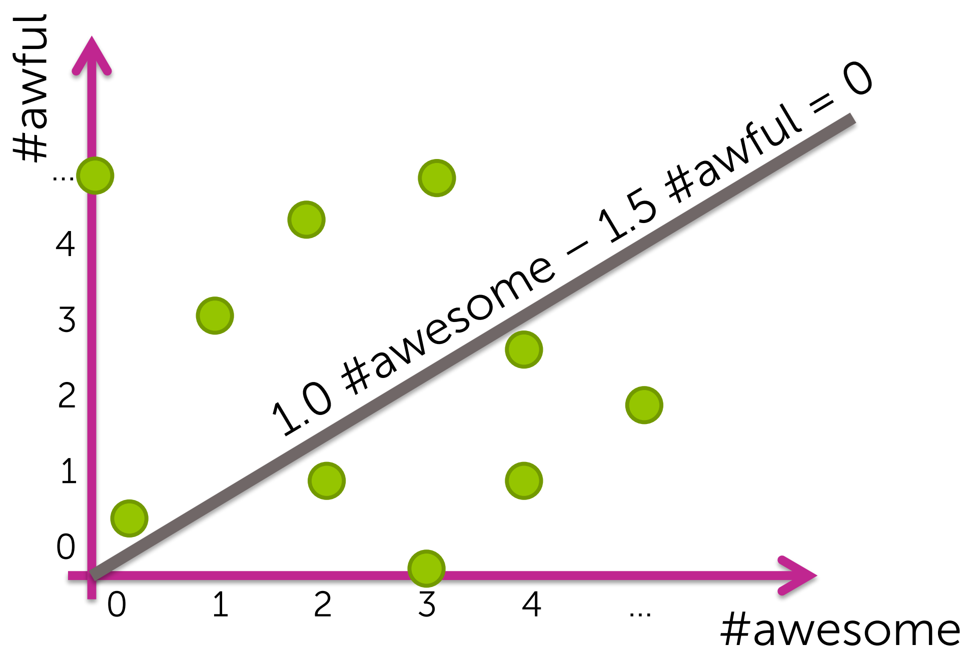
Understanding the logistic regression model

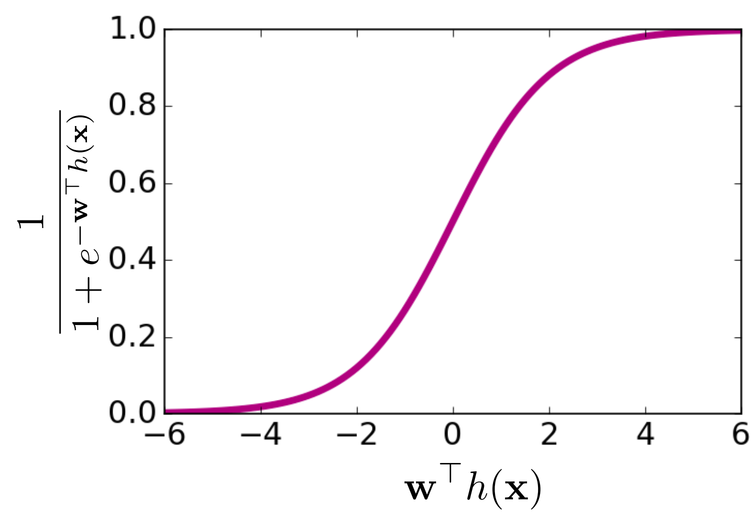
$$P(y=+1 \mid \mathbf{x},\mathbf{w}) = \frac{1}{1+e^{-\mathbf{w}^\top h(\mathbf{x})}}$$



Score($\mathbf{x}_i$)	$P(y=+1 \mid \mathbf{x}_i,\mathbf{w})$

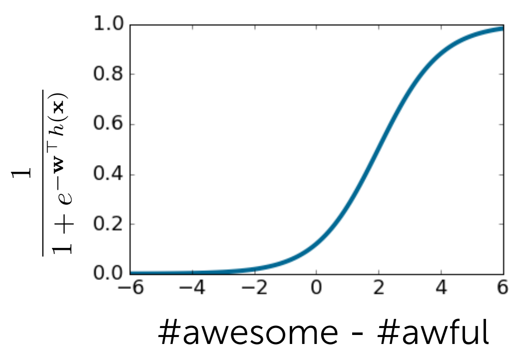
Logistic regression → Linear decision boundary



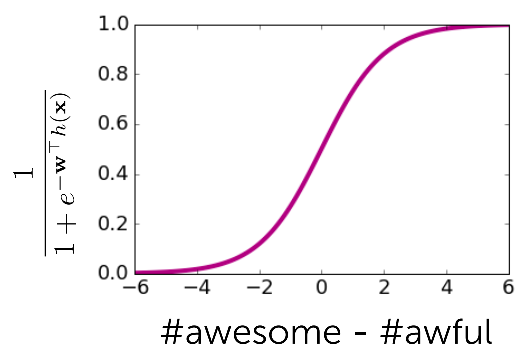


Effect of coefficients on logistic regression model

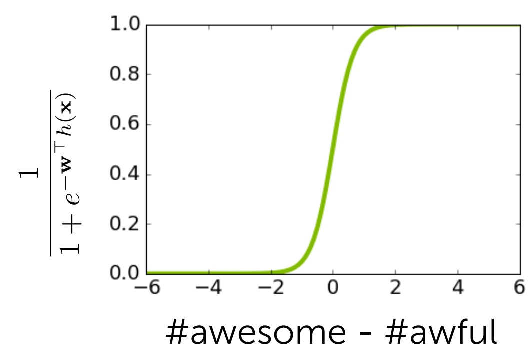
w_0	-2
$w_{\#awesome}$	+1
$w_{\#awful}$	-1



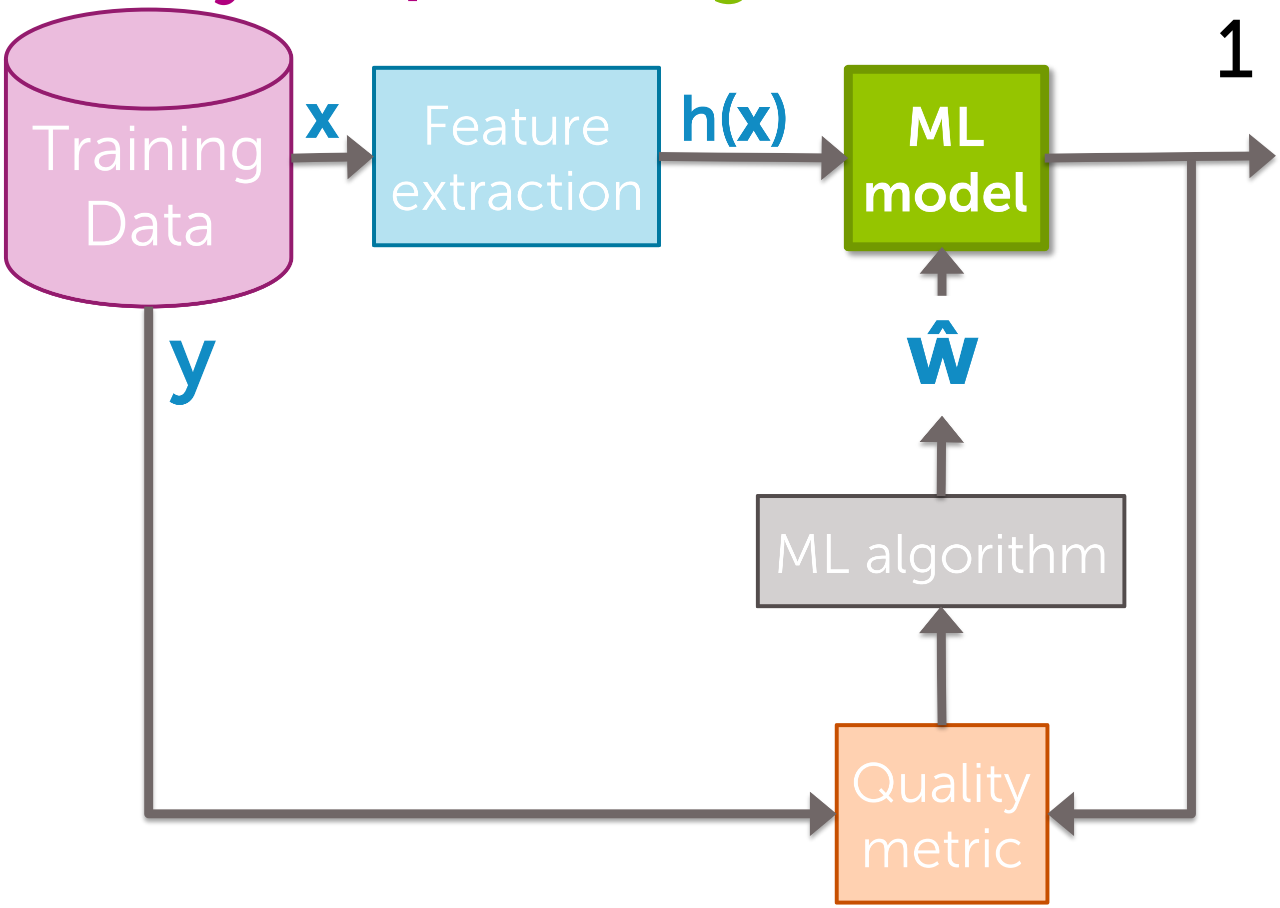
w_0	0
$w_{\#awesome}$	+1
$w_{\#awful}$	-1



w_0	0
$w_{\#awesome}$	+3
$w_{\#awful}$	-3



$$\hat{P}(y=+1|\mathbf{x},\hat{\mathbf{w}}) = \text{sigmoid}(\hat{\mathbf{w}}^T h(\mathbf{x})) = \frac{1}{1+e^{-\hat{\mathbf{w}}^T h(\mathbf{x})}}$$

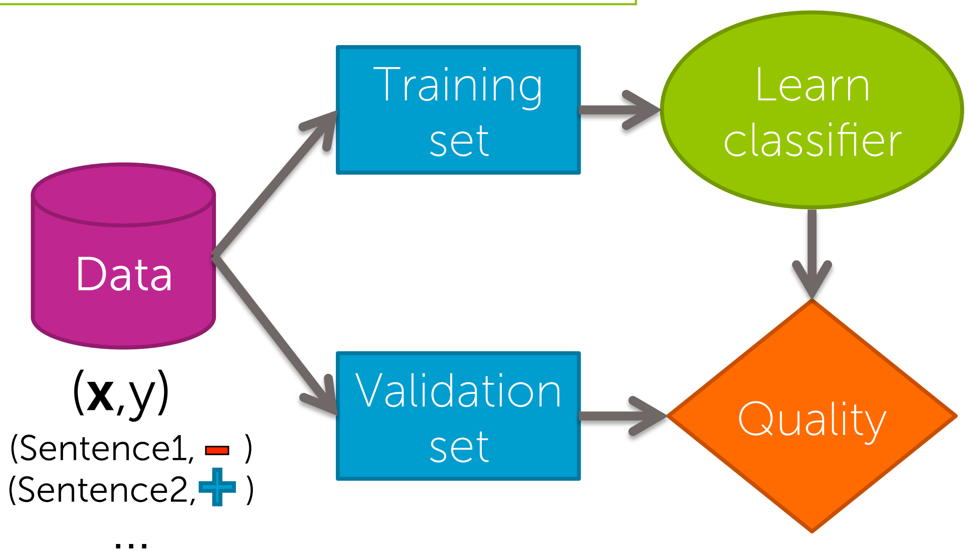


Overview of learning logistic regression model

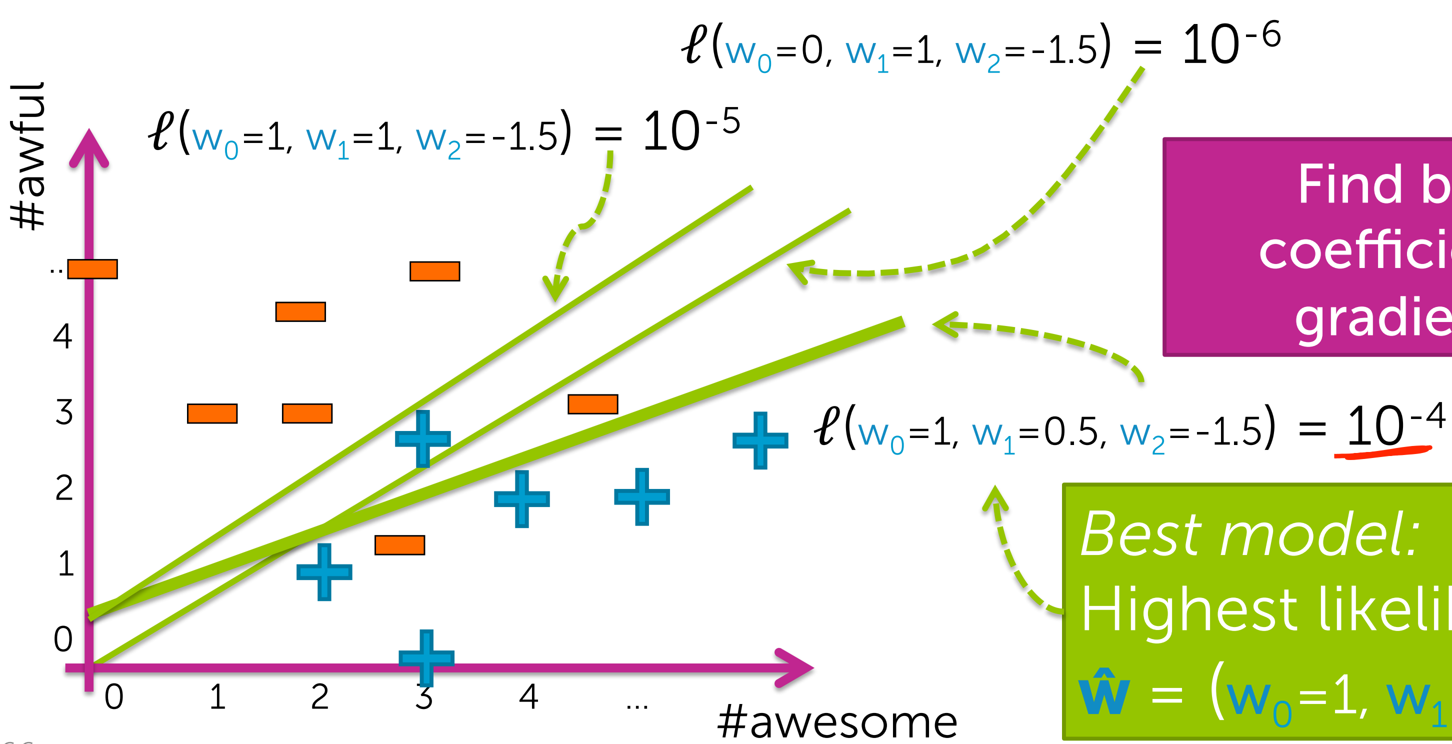
Training a classifier = Learning the coefficients

Word	Coefficient	Value
	$\hat{w}_0$	-2.0
good	$\hat{w}_1$	1.0
awesome	$\hat{w}_2$	1.7
bad	$\hat{w}_3$	-1.0
awful	$\hat{w}_4$	-3.3
…	…	…

$$\hat{P}(y=+1|\mathbf{x},\hat{\mathbf{w}}) = \frac{1}{1+e^{-\hat{\mathbf{w}}^\top h(\mathbf{x})}}$$



Find "best" classifier =
Maximize $\underbrace{\text{quality metric}}_{\text{Likelihood } \ell(\mathbf{w})}$ over all possible w_0, w_1, w_2



Find best model coefficients **w** with gradient ascent!

Best model:
Highest likelihood $\ell(\mathbf{w})$
$\hat{\mathbf{w}} = (w_0=1,\ w_1=0.5,\ w_2=-1.5)$

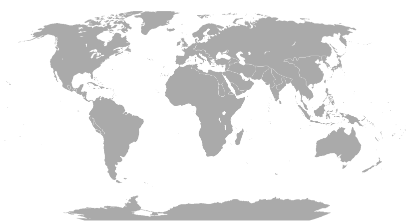
Encoding categorical inputs

Categorical inputs

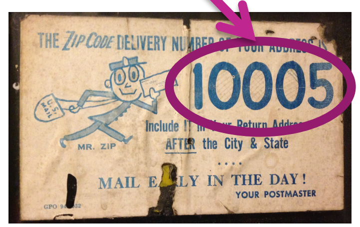
- Numeric inputs:
  - #awesome, age, salary,…
  - Intuitive when multiplied by coefficient
    - e.g., 1.5 #awesome
- Categorical inputs:

Numeric value, but should be interpreted as category
(98195 not about 9x larger than 10005)

Gender
(Male, Female,…)



Country of birth
(Argentina, Brazil, USA,…)

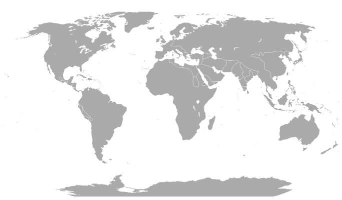


Zipcode
(10005, 98195,…)

How do we multiply category by coefficient???
Must convert categorical inputs into numeric features

Encoding categories as numeric features

x =



Country of birth
(Argentina, Brazil, USA,...)

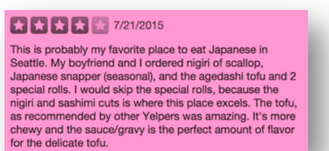
196 categories

1-hot encoding

	Argentina	Brazil	. . .	. . .	Zimbabwe
x	$h_1(\mathbf{x})$	$h_2(\mathbf{x})$	...	$h_{195}(\mathbf{x})$	$h_{196}(\mathbf{x})$
Brazil	0	1	0	0	0
Zimbabwe	0	0	0	0	1

196 features

x =


7/21/2015
This is probably my favorite place to eat Japanese in Seattle. My boyfriend and I ordered nigiri of scallop, Japanese snapper (seasonal), and the agedashi tofu and 2 special rolls. I would skip the special rolls, because the nigiri and sashimi cuts is where this place excels. The tofu, as recommended by other Yelpers was amazing. It's more chewy and the sauce/gravy is the perfect amount of flavor for the delicate tofu.

Restaurant review
(Text data)

10,000 words in vocabulary

Bag of words

	#awesome	#awful	. .	. .	#sushi
x	$h_1(\mathbf{x})$	$h_2(\mathbf{x})$	...	$h_{9999}(\mathbf{x})$	$h_{10000}(\mathbf{x})$
	2	0	0	0	3

10,000 features

Multiclass classification using 1 versus all

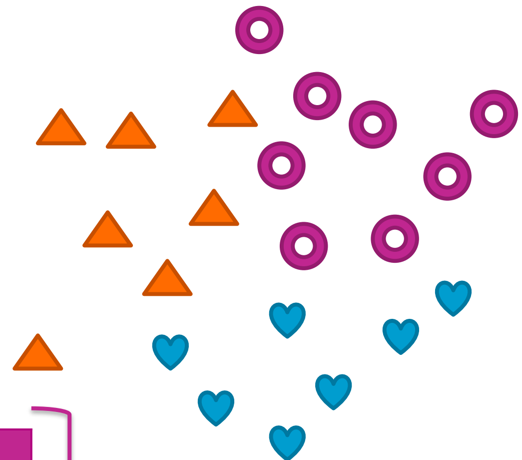
Multiclass classification

Input: x
Image pixels

Output: y
Object in image

Multiclass classification formulation

- C possible classes:
  - y can be 1, 2,…, C
- N datapoints:



Data point	**x**[1]	**x**[2]	y
$\mathbf{x}_1, y_1$	2	1	▲
$\mathbf{x}_2, y_2$	0	2	♥
$\mathbf{x}_3, y_3$	3	3	○
$\mathbf{x}_4, y_4$	4	1	○

Learn:

$\hat{P}(y=$ ▲ $|\mathbf{x})$

$\hat{P}(y=$ ♥ $|\mathbf{x})$

$\hat{P}(y=$ ○ $|\mathbf{x})$

1 versus all:

Estimate $\hat{P}(y=\triangle|\mathbf{x})$ using 2-class model

+1 class: points with y_i= △

-1 class: points with y_i= ♥ OR ○

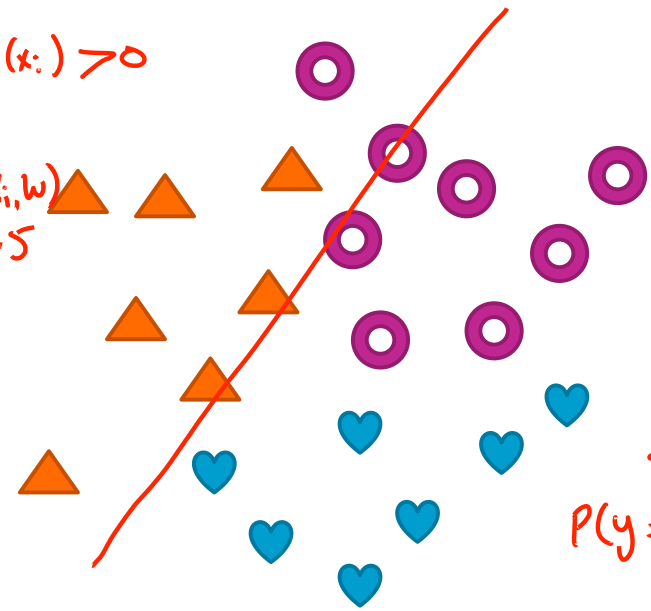
Train classifier: $\hat{P}_{\triangle}(y=+1|\mathbf{x})$

Predict: $\hat{P}(y=\triangle|\mathbf{x}_i) = \hat{P}_{\triangle}(y=+1|\mathbf{x}_i)$

Score$(x_i) > 0$

⇓

$P(y=\triangle|x_i,w) > 0.5$



Score$(x_i) < 0$

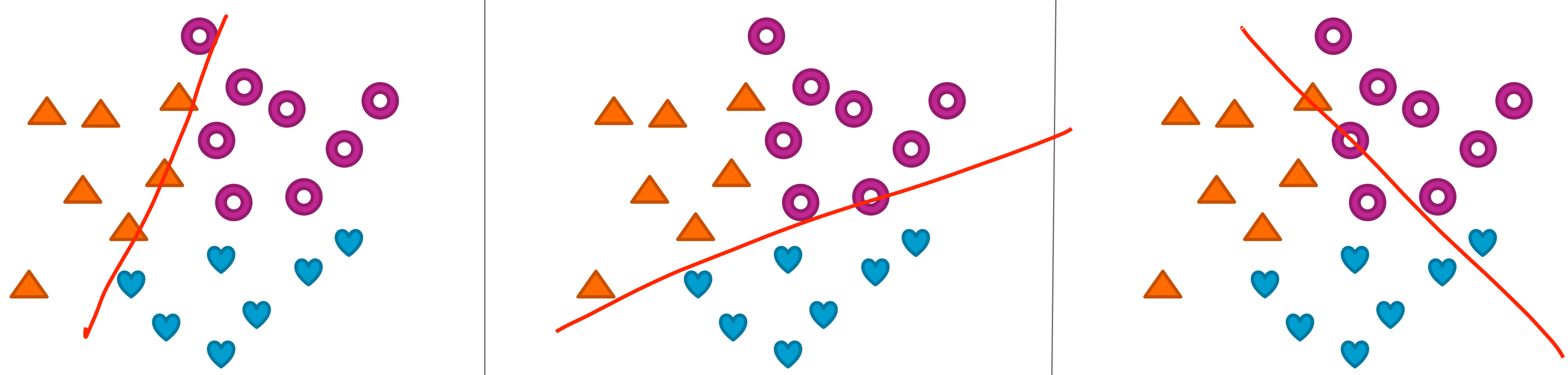
$P(y=\triangle|x_i,w) < 0.5$

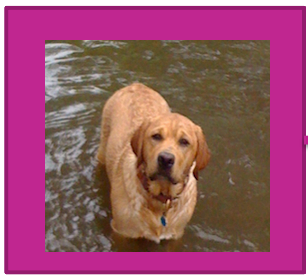
1 versus all: simple multiclass classification using *C* 2-class models

$\hat{P}(y=\triangle|\mathbf{x}_i) = \hat{P}_{\triangle}(y=+1|x_i, w_{\triangle})$

$\hat{P}(y=\heartsuit|\mathbf{x}_i) = \hat{P}_{\heartsuit}(y=+1|x_i, w_{\heartsuit})$

$\hat{P}(y=\circledcirc|\mathbf{x}_i) = \hat{P}_{\circledcirc}(y=+1|x_i, w_{\circledcirc})$





Input: $\mathbf{x}_i$

Multiclass training

$\hat{P}_c(y=+1|\mathbf{x})$ = estimate of 1 vs all model for each class

Predict most likely class

max_prob = 0; $\hat{y}$ = 0

For c = 1,…,C:

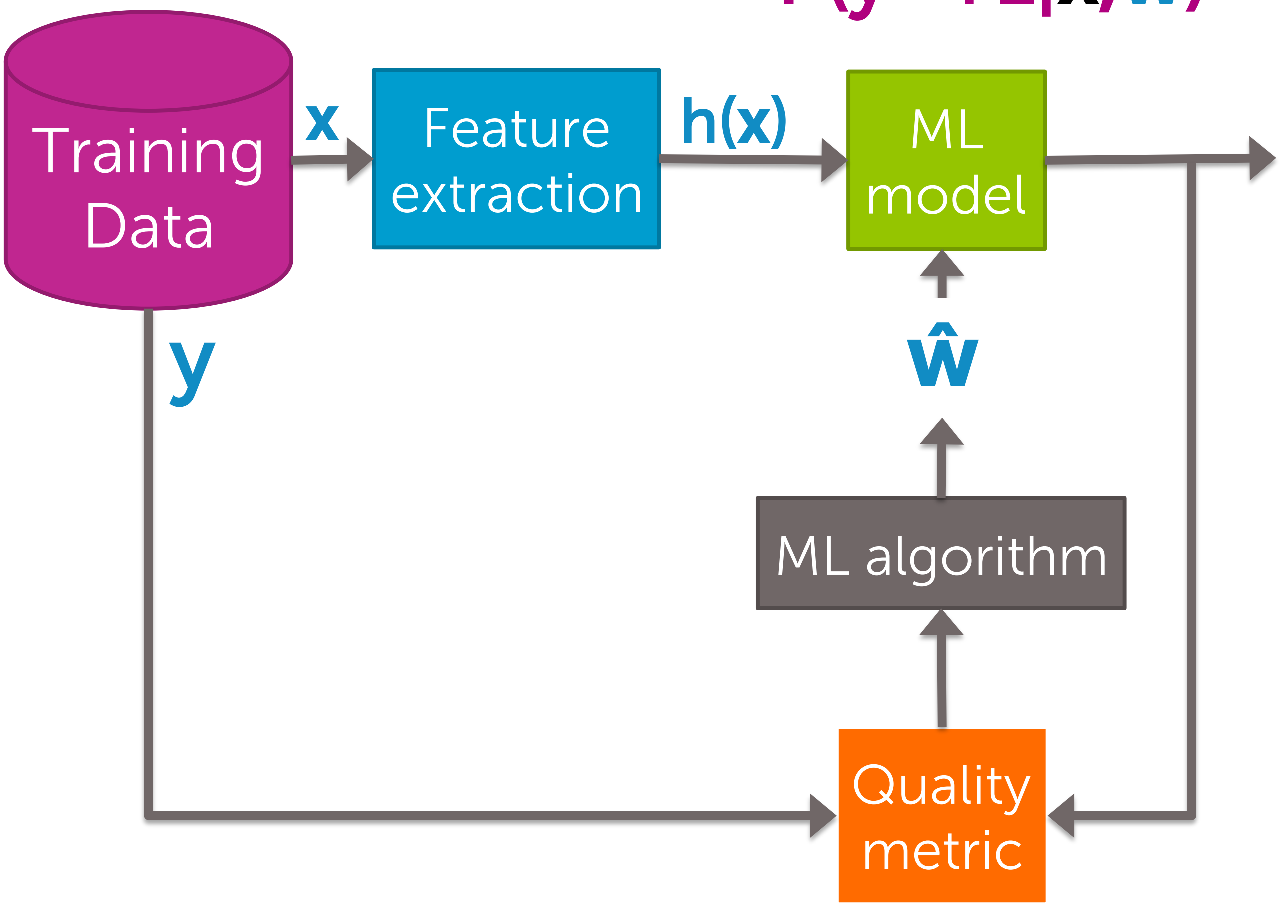
  If $\hat{P}_c(y=+1|\mathbf{x}_i) >$ max_prob:

    $\hat{y}$ = c

    max_prob = $\hat{P}_c(y=+1|\mathbf{x}_i)$

Summary of logistic regression classifier

$$\hat{P}(y=+1|\mathbf{x},\hat{\mathbf{w}}) = \frac{1}{1+e^{-\hat{\mathbf{w}}^\top h(\mathbf{x})}}$$



What you can do now…

- Describe decision boundaries and linear classifiers
- Use class probability to express degree of confidence in prediction
- Define a logistic regression model
- Interpret logistic regression outputs as class probabilities
- Describe impact of coefficient values on logistic regression output
- Use 1-hot encoding to represent categorical inputs
- Perform multiclass classification using the 1-versus-all approach