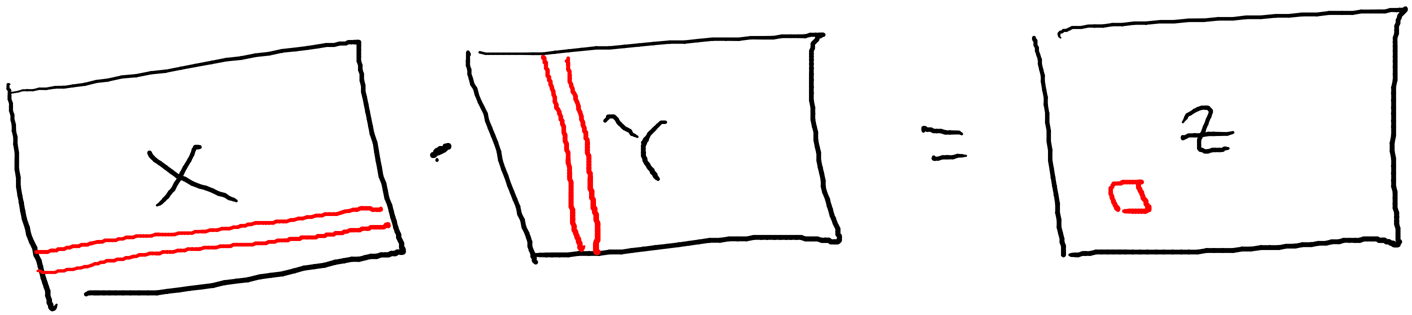
Design and Analysis of Algorithms I

Divide and Conquer

Matrix Multiplication

Matrix Multiplication



(all n X n matrices)

Where z_{ij} = (i^{th} row of X) . (j^{th} column of Y)

$= \sum_{k=1}^{n} X_{ik} \cdot Y_{kj}$

<u>Note</u> : input size

= $\theta(n^2)$

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Example (n=2)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

$$z_{ij} = \sum_{k=1}^{n} X_{ik} \cdot Y_{kj}$$

$$\theta(n)$$

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What is the asymptotic running time of the straightforward iterative algorithm for matrix multiplication?

- $\theta(n \log n)$
- $\theta(n^2)$
- $\theta(n^3)$
- $\theta(n^4)$

The Divide and Conquer Paradigm

1. DIVIDE into smaller subproblems
2. CONQUER subproblems recursively.
3. COMBINE solutions of subproblems into one for the original problem.

Applying Divide and Conquer

<u>Idea</u> :

Write $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $Y = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$

[where A through H are all n/2 by n/2 matrices]

<u>Then</u> : (you check)

$$X \cdot Y = \begin{pmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{pmatrix}$$

Recursive Algorithm #1

Step 1 : recursively compute the 8 necessary products.

Step 2 : do the necessary additions $(\theta(n^2)\ time)$

Fact : runtime is $\theta(n^3)$ [follows from the master method]

Strassen's Algorithm (1969)

Step 1 : recursively compute only 7 (cleverly chosen) products

Step 2 : do the necessary (clever) additions + subtractions (still $\theta(n^2)$ time)

Fact : better than cubic time!

[see Master Method lecture]

$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ $Y = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$

The Details

The Seven Products : $P_1 = A(F-H)$, $P_2 = (A+B)H$, $P_3 = (C+D)E$, $P_4 = D(G-E)$, $P_5 = (A+D)(E+H)$, $P_6 = (B-D)(G+H)$, $P_7 = (A-C)(E+F)$

Claim : $X \cdot Y = \begin{pmatrix} AE+BG & AF+BH \\ CE+DG & CF+DH \end{pmatrix} = \begin{pmatrix} P_5+P_4-P_2+P_6 & P_1+P_2 \\ P_3+P_4 & P_1+P_5-P_3-P_7 \end{pmatrix}$

Proof: $AE + \cancel{AH} + \cancel{DE} + \cancel{DH} + \cancel{DG} - \cancel{DE} - \cancel{AH} - \cancel{BH} + BG + \cancel{BH} - \cancel{DG} - \cancel{DH} = AE + BG$

Q.E.D

Question : where did this come from ? (remains open!)