Design and Analysis of Algorithms I

Graph Primitives

Dijkstra's Algorithm: Why It Works

Dijkstra's Algorithm

This array only to help explanation!

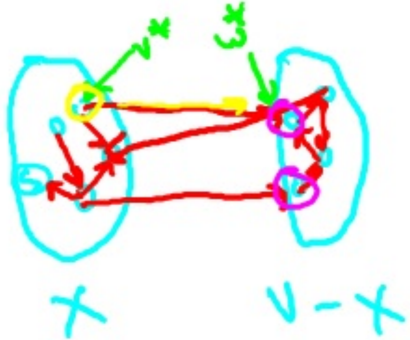
<u>Initialize</u>:

- X = [s] [vertices processed so far]
- A[s] = 0 [computed shortest path distances]
- B[s] = empty path [computed shortest paths]

<u>Main Loop</u>

- while X≠V:

-need to grow x by one node



<u>Main Loop cont'd:</u>

- among all edges $(v, w) \in E$ with $v \in X, w \notin X$, pick the one that minimizes $A[v] + l_{vw}$

Already computed in earlier iteration

[call it (v*, w*)]

- add w* to X
- set $A[w^*] := A[v^*] + l_{v^*w^*}$
- set $B[w^*] := B[v^*]u(v^*, w^*)$

Correctness Claim

Theorem [Dijkstra] For every directed graph with nonnegative edge lengths, Dijkstra's algorithm correctly computes all shortest-path distances.

$$[i.e., \;\; A[v] = L(v) \;\forall v \in V]$$

what algorithm computes

True shortest distance from s to v

Proof: by induction on the number of iterations.

Base Case: $A[s] = L[s] = 0$ (correct)

Proof

Inductive Step:

Inductive Hypothesis: all previous iterations correct (i.e., $A[v] = L(v)$ and $B[v]$ is a true shortest s-v path in G, for all v already in X).

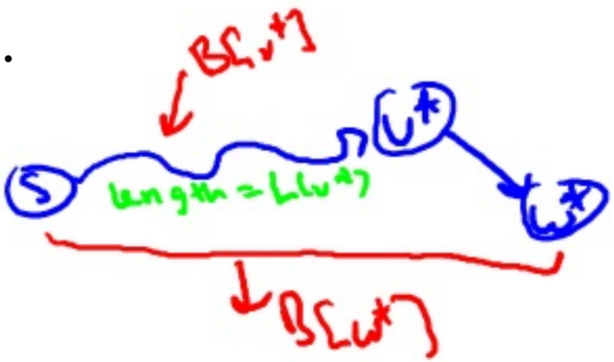
In current iteration:

in X   not in X

We pick an edge (v^*, w^*) and we add w^* to X.

We set $B[w^*] = B[v^*]\ u(v^*, w^*)$

has length $L(v^*) + l_{v^*w^*}$ has length $L(v^*)$



L(v*) by I.H

Also: $A[w^*] = A[v^*] + l_{v^*w^*} = L(v^*) + l_{v^*w^*}$

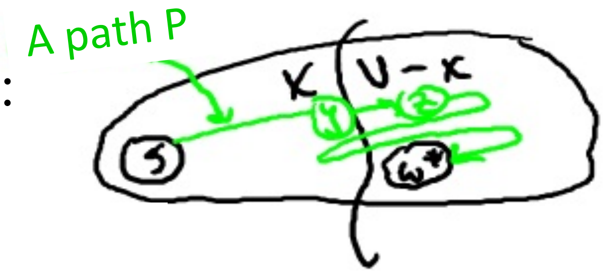
Proof (con'd)

Upshot: in current iteration, we set:

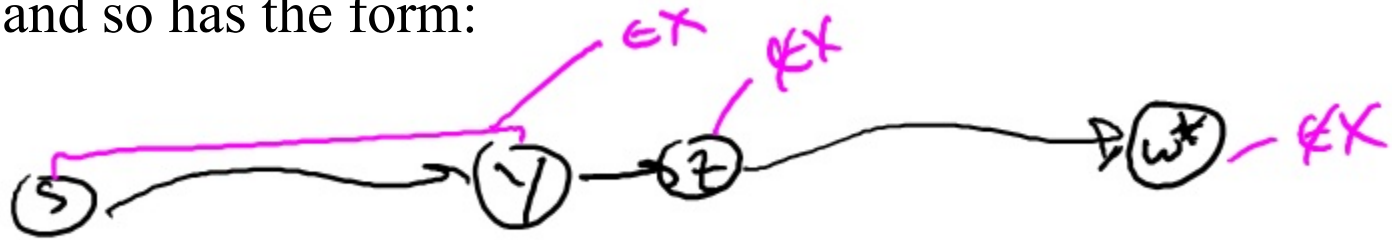
1. $A[w^*] = L(v^*) + l_{v^*w^*}$
2. $B[w^*]$ = an s -> w* path with length $(L(v^*) + l_{v^*w^*})$

To finish proof: need to show that *every* s-w* path has length >= $L(v^*) + l_{v^*w^*}$ (if so, our path is the shortest!)

So: Let P= any s->w* path. Must "cross the frontier":

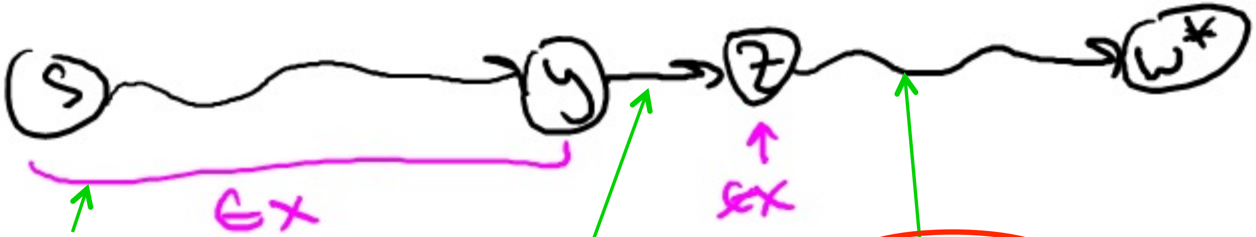


and so has the form:



Proof (con'd)

So: every s->w* path P has to have the form



(Since no negative edges !)

>= length of shortest s-y path = L(y) = A[y]

Length = l_{yz}

Length >=0

By inductive hypothesis (since $y \in X$)

($y \in X$
$z \notin X$)

Total length of path P: at least A[y] + C_{yz}

length of our path !

-> by Dijkstra's greedy criterion, $A[v^*] + l_{v^* w^*} \leq A[y] + l_{yz} \leq$ length of P

Q.E.D.