Design and Analysis of Algorithms I

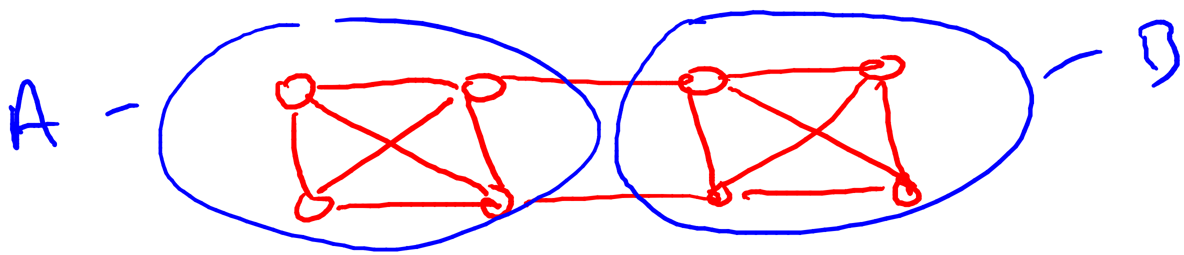
# Contraction Algorithm

# The Analysis

# The Minimum Cut Problem

<u>Input</u>: An undirected graph G = (V, E).
[parallel edges  allowed]
[See other video for representation of input]

<u>Goal</u>: Compute a cut with fewest number of crossing edges.
(a <u>min cut</u>)



# Random Contraction Algorithm

[ due to Karger, early 90s]

While there are more than 2 vertices:

- pick a remaining edge (u,v) uniformly at random
- merge (or "contract" ) u and v into a single vertex
- remove self-loops

return cut represented by final 2 vertices.

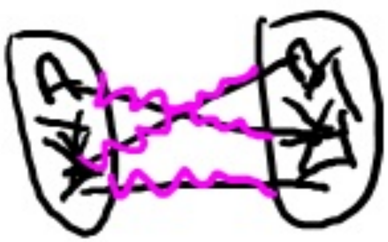
# The Setup

Question: what is the probability of success?

Fix a graph G = (V, E) with n vertices, m edges.

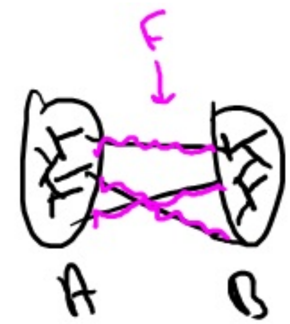
Fix a minimum cut (A, B).

Let k = # of edges crossing (A, B). (Call these edges F)



# What Could Go Wrong?

1. Suppose an edge of F is contracted at some point
   $\Rightarrow$ algorithm will not output (A,B).




2. Suppose only edges inside A or inside B get contracted $\Rightarrow$ algorithm will output (A, B).

Thus: Pr [ output is (A, B) ] = Pr [ never contracts an edge of F]

Let $S_i$ = event that an edge of F contracted in iteration i.

Goal: Compute $\Pr[\neg S_1 \wedge \neg S_2 \wedge \neg S_3 \wedge \ldots\ldots \wedge \neg S_{n-2}]$

What is the probability that an edge crossing the minimum cut $(A, B)$ is chosen in the first iteration (as a function of the number of vertices $n$, the number of edges $m$, and the number $k$ of crossing edges)?

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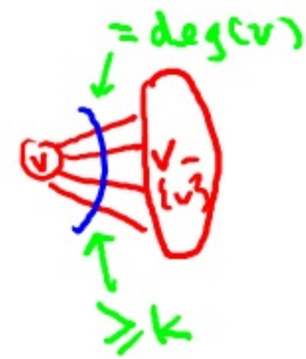
- $k/n$
- $k/m$
- $k/n^2$
- $n/m$

$$\Pr[S_1] = \frac{\#\text{ of crossing edges}}{\#\text{ of edges}} = \frac{k}{m}$$

# The First Iteration

<u>Key Observation</u>: degree of each vertex is at least k

**# of incident edges**




<u>Reason</u>: each vertex v defines a cut ({v}. V-{v}).

Since $\underbrace{\sum_v degree(v)}_{\geq kn} = 2m$, we have $m \geq \frac{kn}{2}$

Since $\Pr[S_1] = \frac{k}{m}$, $\Pr[S_1] \leq \frac{2}{n}$

# The Second Iteration

Recall: $\Pr[\neg S_1 \wedge \neg S_2] = \underbrace{\Pr[\neg S_2 | \neg S_1]}_{= 1 - \frac{k}{\text{\# of remaining edge}}} . \Pr[\neg S_1]$

$\geq (1 - \frac{2}{n})$

what is this?

Note: all nodes in contracted graph define cuts in G (with at least k crossing edges).

➢ all degrees in contracted graph are at least k

So: # of remaining ec $\geq \frac{1}{2}k(n-1)$

So $\Pr[\neg S_2 | \neg S_1] \geq 1 - \frac{2}{(n-1)}$

# All Iterations

In general:

$$\Pr[\neg S_1 \wedge \neg S_2 \wedge \neg S_3 \wedge .... \wedge \neg S_{n-2}]$$

$$= \Pr[\neg S_1] \Pr[\neg S_2 | \neg S_1] \Pr[\neg S_3 | \neg S_2 \wedge \neg S_1] ...... \Pr[\neg S_{n-2} | \neg S_1 \wedge ... \wedge \neg S_{n-3}]$$

$$\geq (1 - \tfrac{2}{n})(1 - \tfrac{2}{n-1})(1 - \tfrac{2}{n-2}).....(1 - \tfrac{2}{n-(n-4)})(1 - \tfrac{2}{n-(n-3)})$$

$$= \frac{\cancel{n-2}}{n} \cdot \frac{\cancel{n-3}}{n-1} \cdot \frac{\cancel{n-4}}{\cancel{n-2}} ...... \frac{2}{\cancel{4}} \cdot \frac{1}{\cancel{3}} = \frac{2}{n(n-1)} \geq \frac{1}{n^2}$$

Problem: low success probability! (But: non trivial)

recall $\simeq 2^n$ cuts !

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# Repeated Trials

Solution: run the basic algorithm a large number N times, remember the smallest cut found.

Question: how many trials needed?

Let $T_i$ = event that the cut (A, B) is found on the $i^{th}$ try.

- by definition, different $T_i$'s are independent

So: $$\Pr[\text{all N trails fail}] = \Pr[\neg T_1 \wedge \neg T_2 \wedge \ldots \wedge \neg T_N]$$

$$= \prod_{i=1}^{N} \Pr[\neg T_i] \le (1 - \frac{1}{n^2})^N$$

**By independence !**

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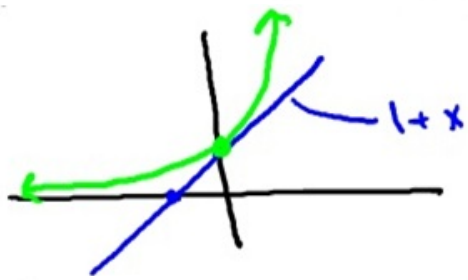
# Repeated Trials (con'd)

Pr[all trials fail] $\leq (1-\frac{1}{n^2})^N$

Calculus fact: $\forall$real numbers x, 1+x $\leq e^x$




So: if we take $N = n^2, \Pr[\text{all fail}] \leq (e^{-\frac{1}{n^2}})^{n^2} = \frac{1}{e}$

If we take $N = n^2 \ln n, \Pr[\text{all fail}] \leq (\frac{1}{e})^{\ln n} = \frac{1}{n}$

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Running time: polynomial in n and m but slow $(\Omega(n^2m))$

But: can get big speed ups ( to roughly $O(n^2)$) with more ideas.