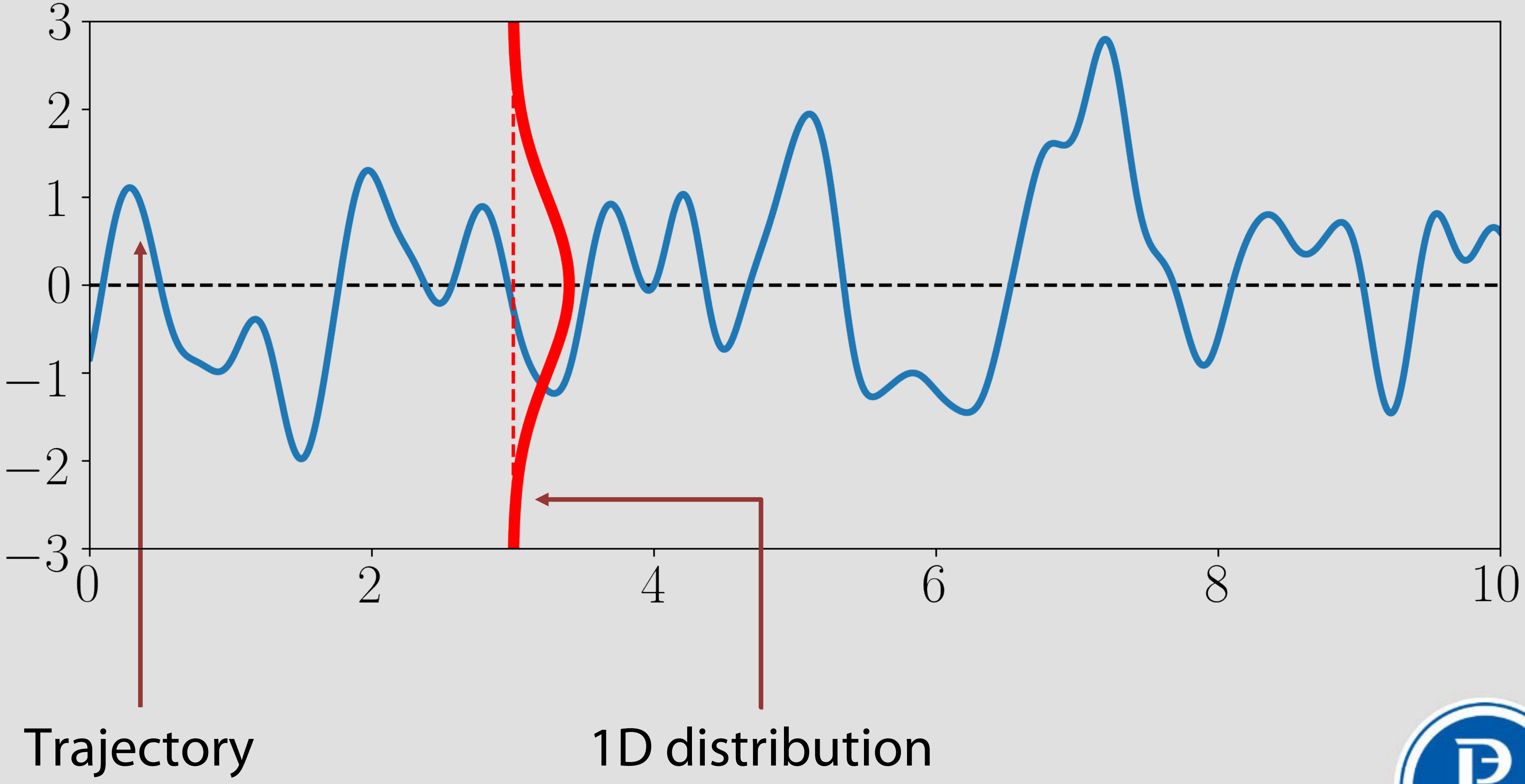
Gaussian processes

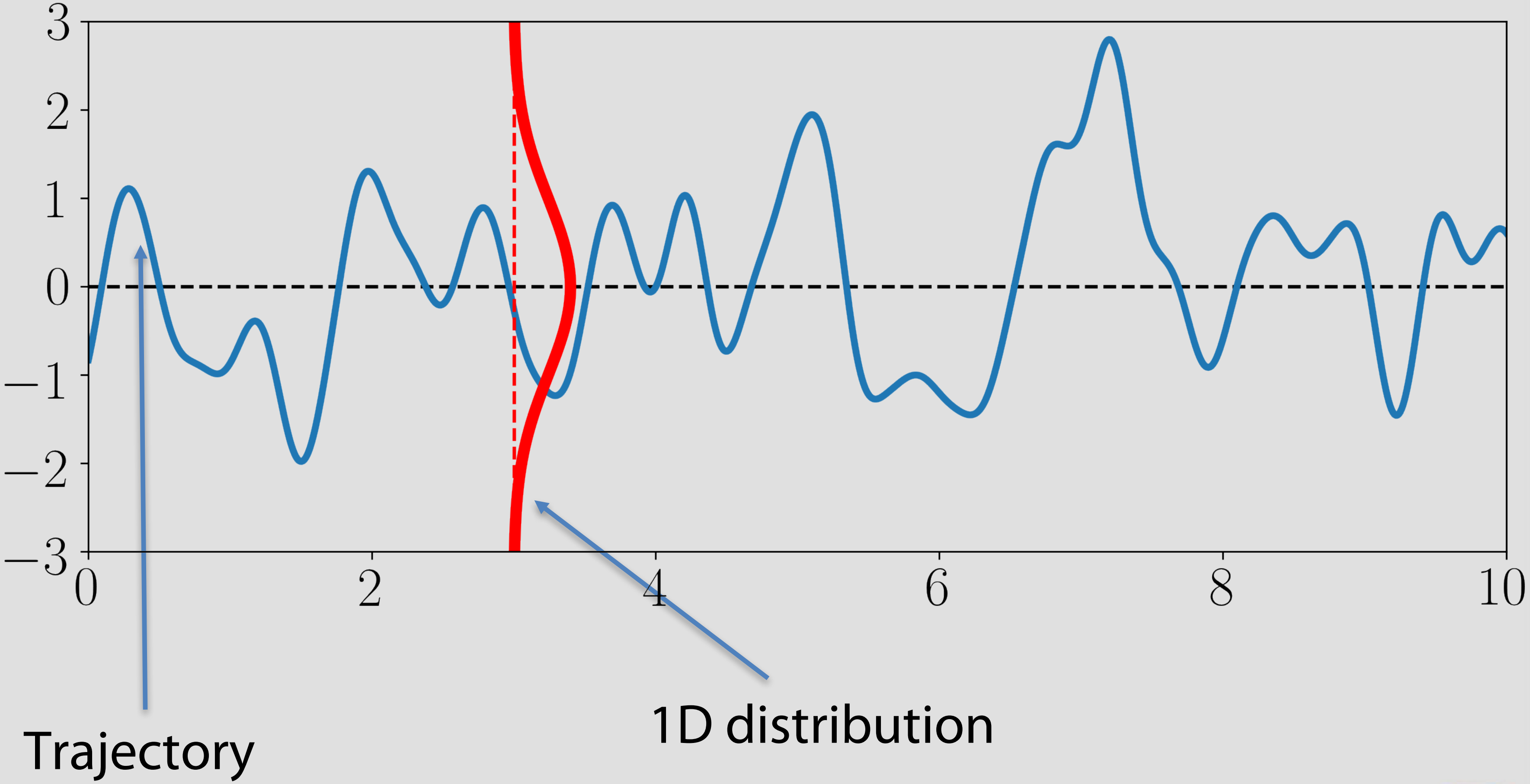
Random process

For any $x \in \mathbb{R}^d$ assign random variable $f(x)$



Random process (Технический слайд)

For any $x \in \mathbb{R}^d$ assign random variable $f(x)$



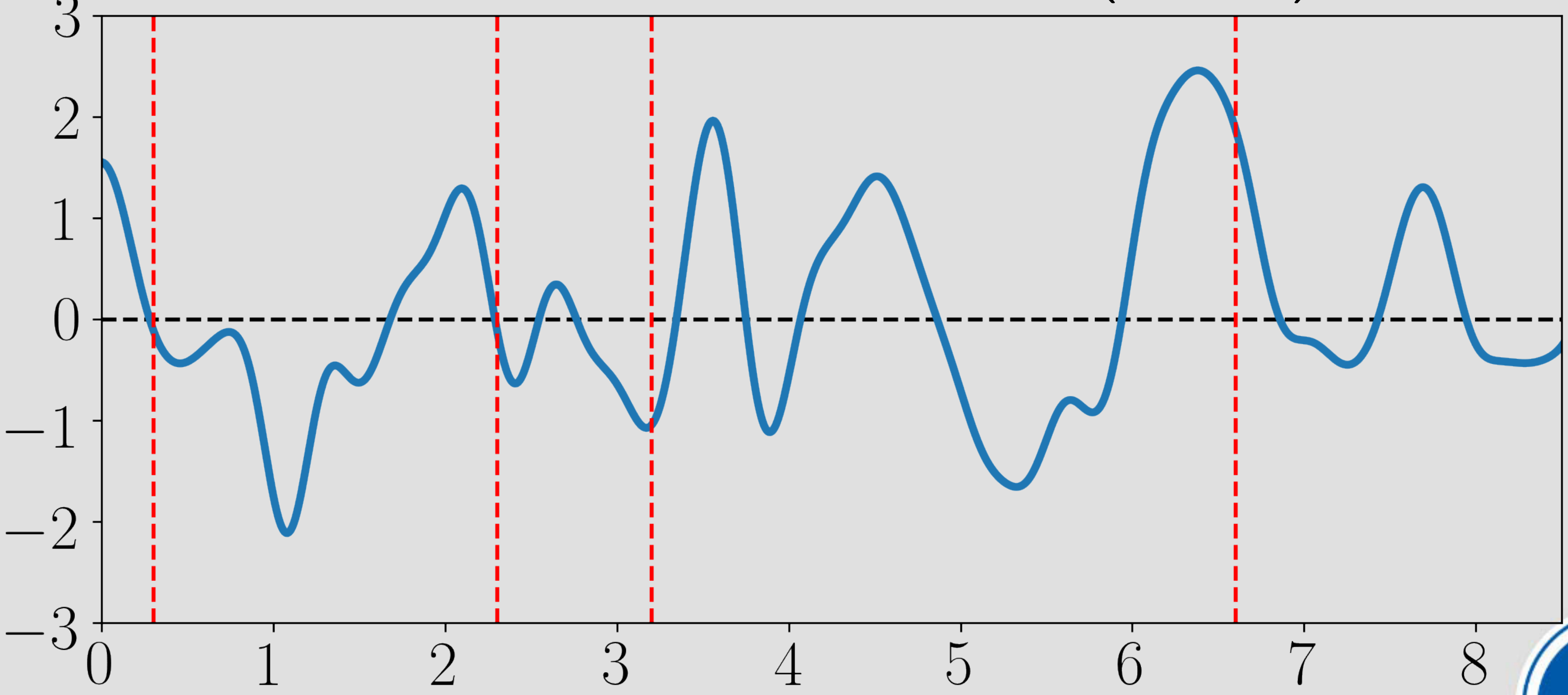
If d=1 => time

Gaussian process

Definition:

Random process f is **Gaussian**, if for any finite number points, their joint distribution is normal:

$$\forall n \in \mathbb{N} \quad \forall x_1, x_2, \ldots, x_n \in \mathbb{R}^d \quad \begin{pmatrix} f(x_1) \\ f(x_2) \\ \ldots \\ f(x_n) \end{pmatrix} \sim \mathcal{N}$$



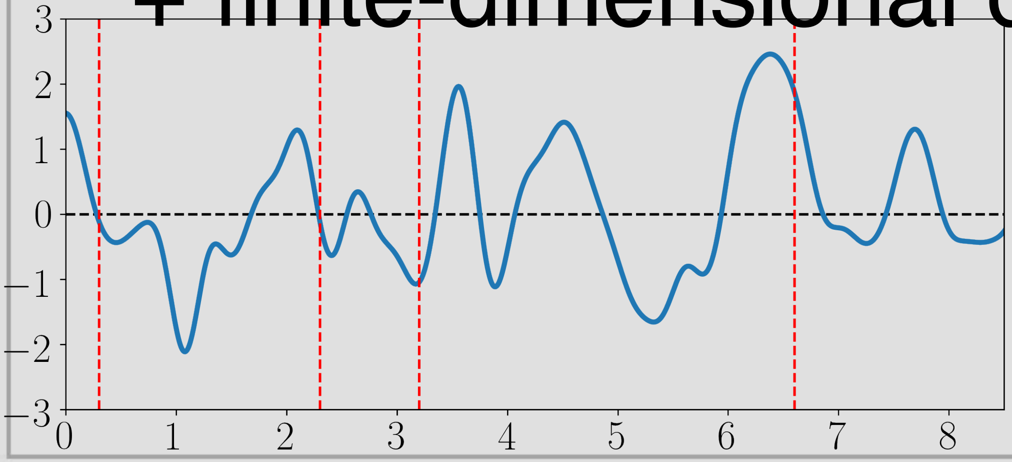
Gaussian process (ТЕХНИЧЕСКИЙ СЛАЙД)

<u>Definition</u>:

Random process f is **Gaussian**, if for any finite number of points, their joint distribution is normal:

$$\forall n \in \mathbb{N} \quad \forall x_1, x_2, \ldots, x_n \in \mathbb{R}^d \quad \begin{pmatrix} f(x_1) \\ f(x_2) \\ \ldots \\ f(x_n) \end{pmatrix} \sim \mathcal{N}$$

Want: joint over all points is normal
+ efficient sampling (plot w/ sampling)
+ finite-dimensional distribution



Gaussian process

$$\forall n \in \mathbb{N} \quad \forall x_1, x_2, \ldots, x_n \in \mathbb{R}^d \ \begin{pmatrix} f(x_1) \\ f(x_2) \\ \ldots \\ f(x_n) \end{pmatrix} \sim \mathcal{N}$$

Parameters:

$$\mathbb{E}f(x) = m(x)$$

$$\mathrm{Cov}[f(x_1), f(x_2)] = K(x_1, x_2)$$

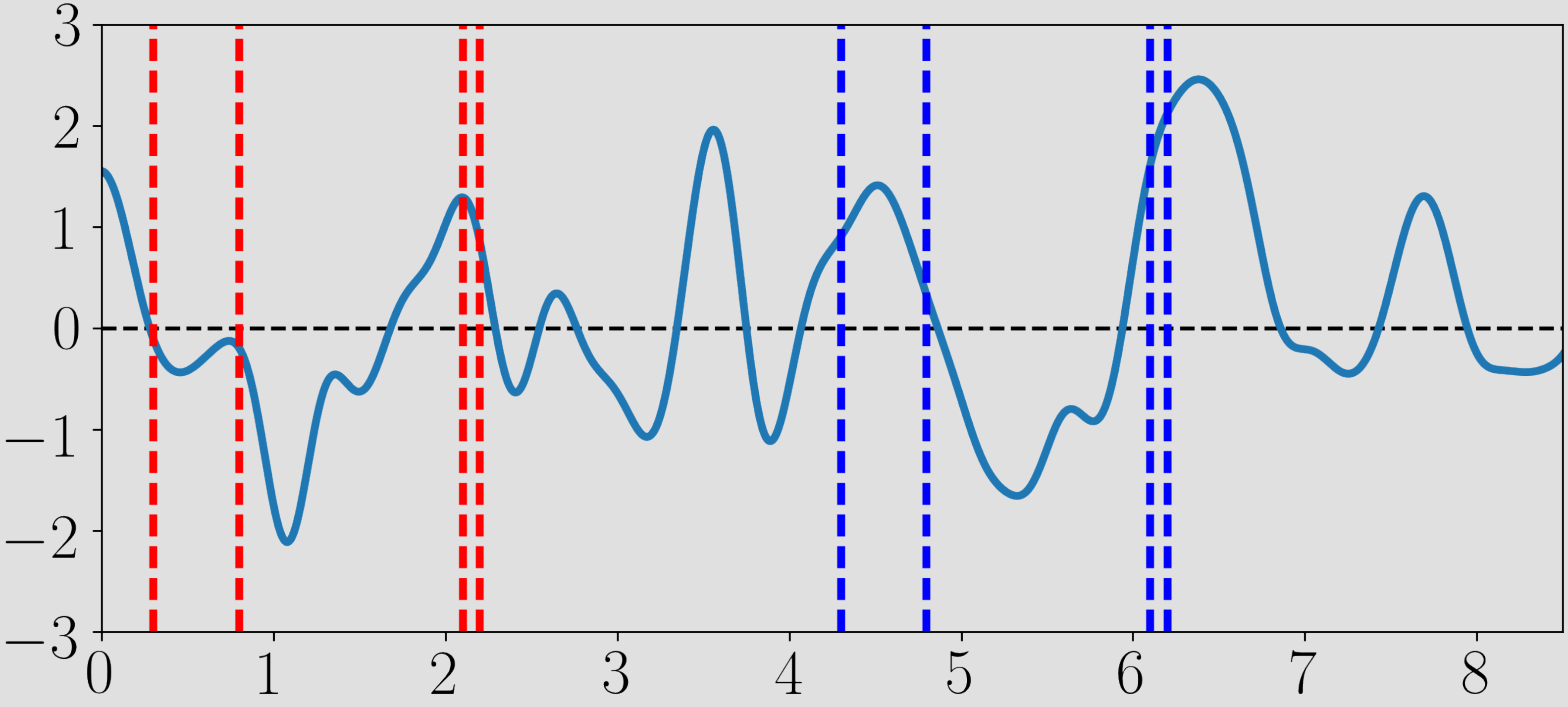
Finally:

$$\begin{pmatrix} f(x_1) \\ f(x_2) \\ \ldots \\ f(x_n) \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} m(x_1) \\ m(x_2) \\ \ldots \\ m(x_n) \end{pmatrix}, \begin{pmatrix} K(x_1,x_1) & \ldots & K(x_1,x_n) \\ K(x_2,x_1) & \ldots & K(x_2,x_n) \\ \ldots & & \ldots \\ K(x_n,x_1) & \ldots & K(x_n,x_n) \end{pmatrix}\right)$$

Stationary process

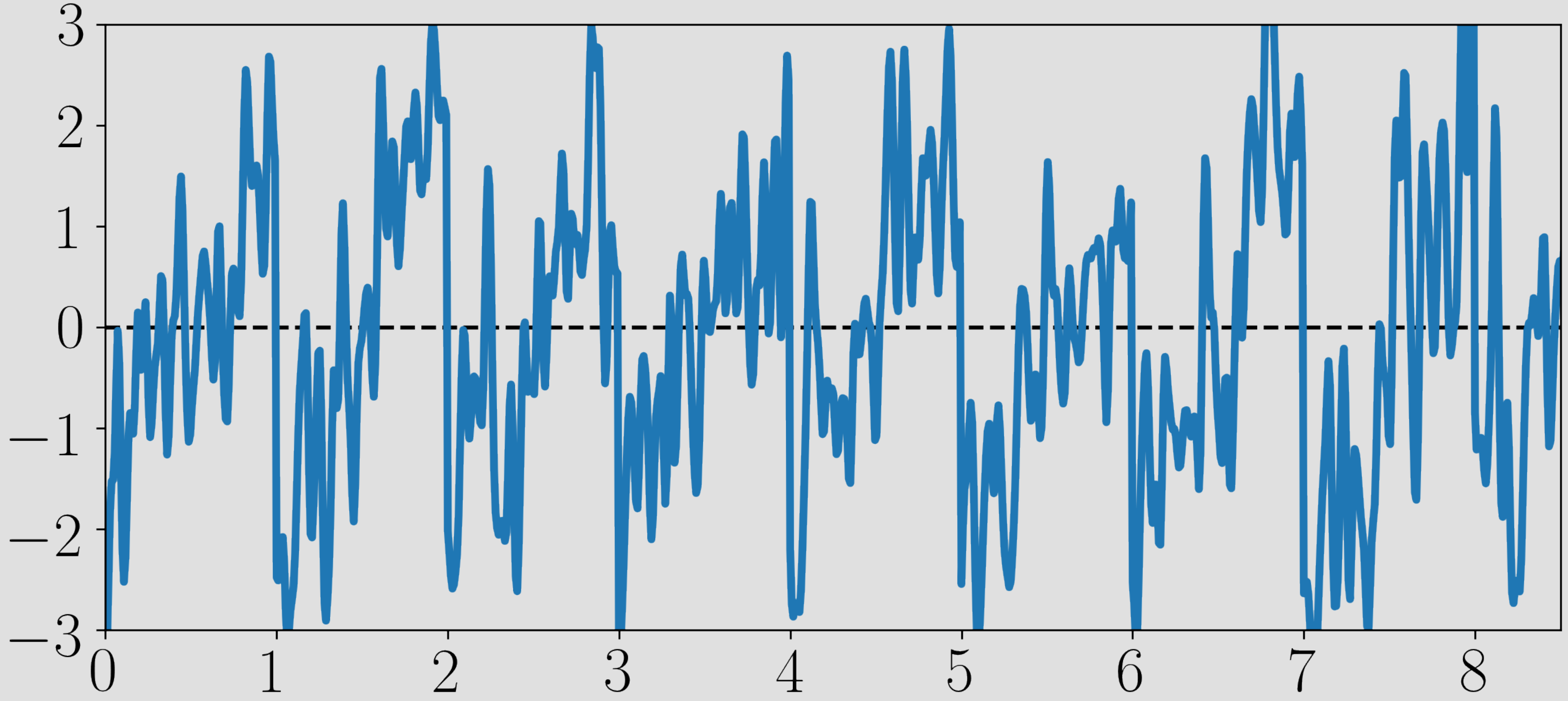
Definition:

Random process f is stationary, if its finite-dimensional distributions depend only on relative position of the points



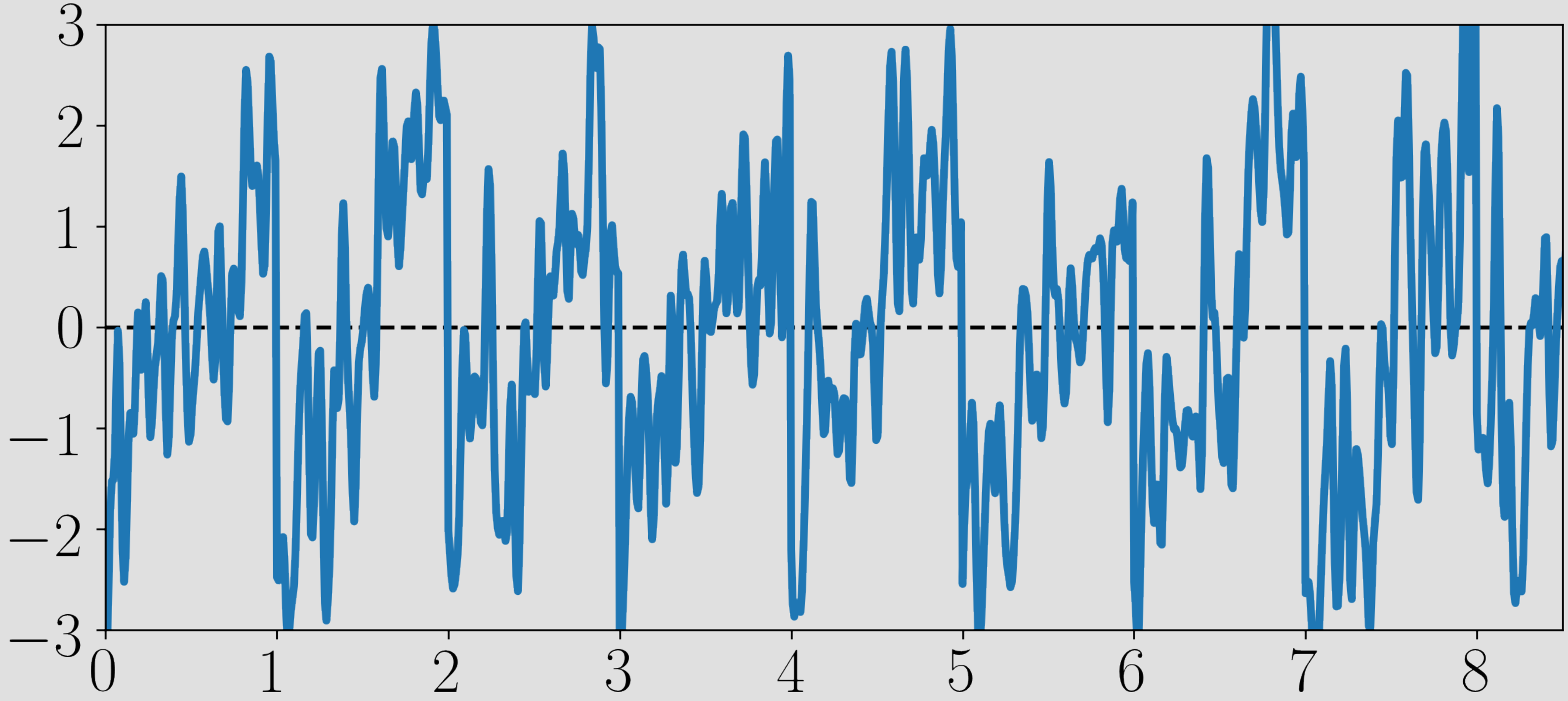
Stationary process

Stationary or not?



Stationary process

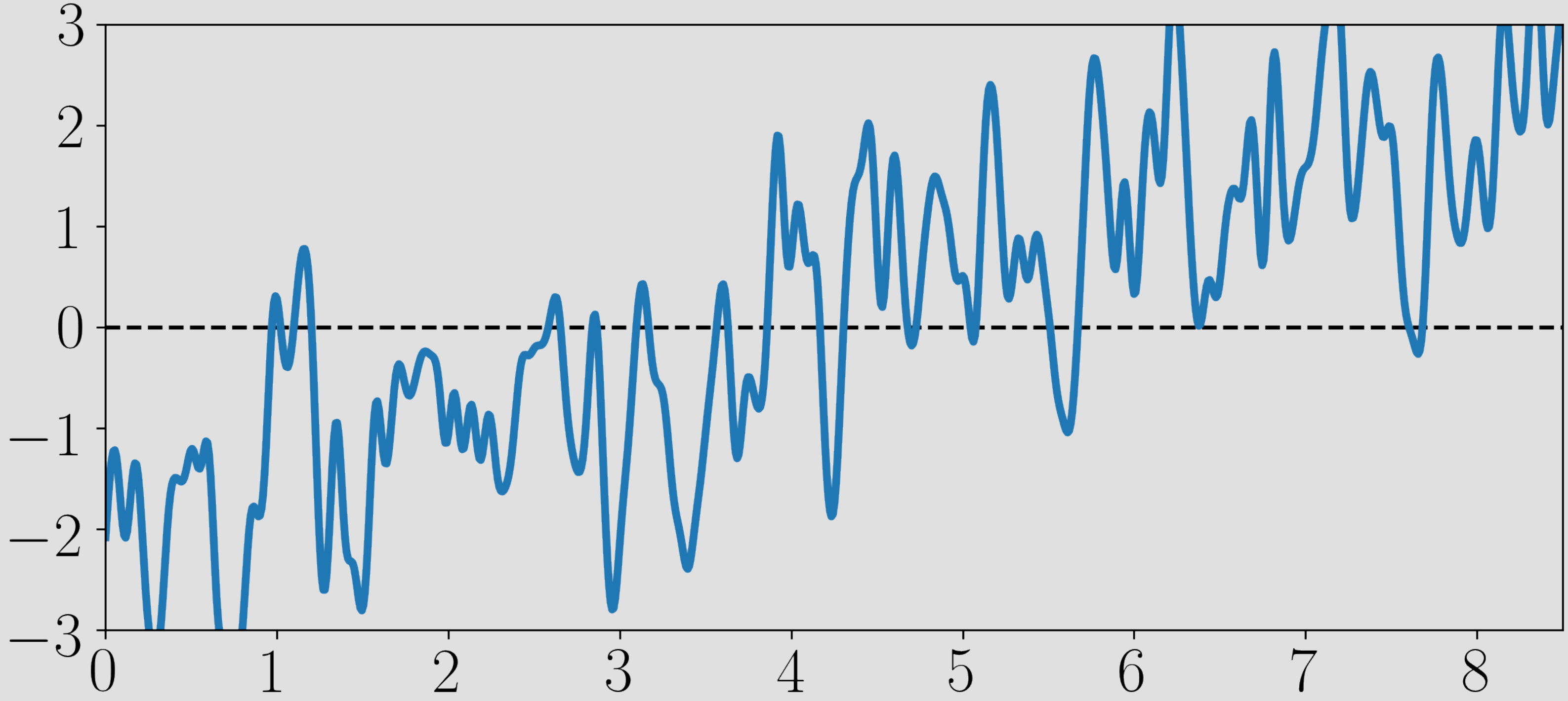
Stationary or not?



Not stationary: **seasonality**

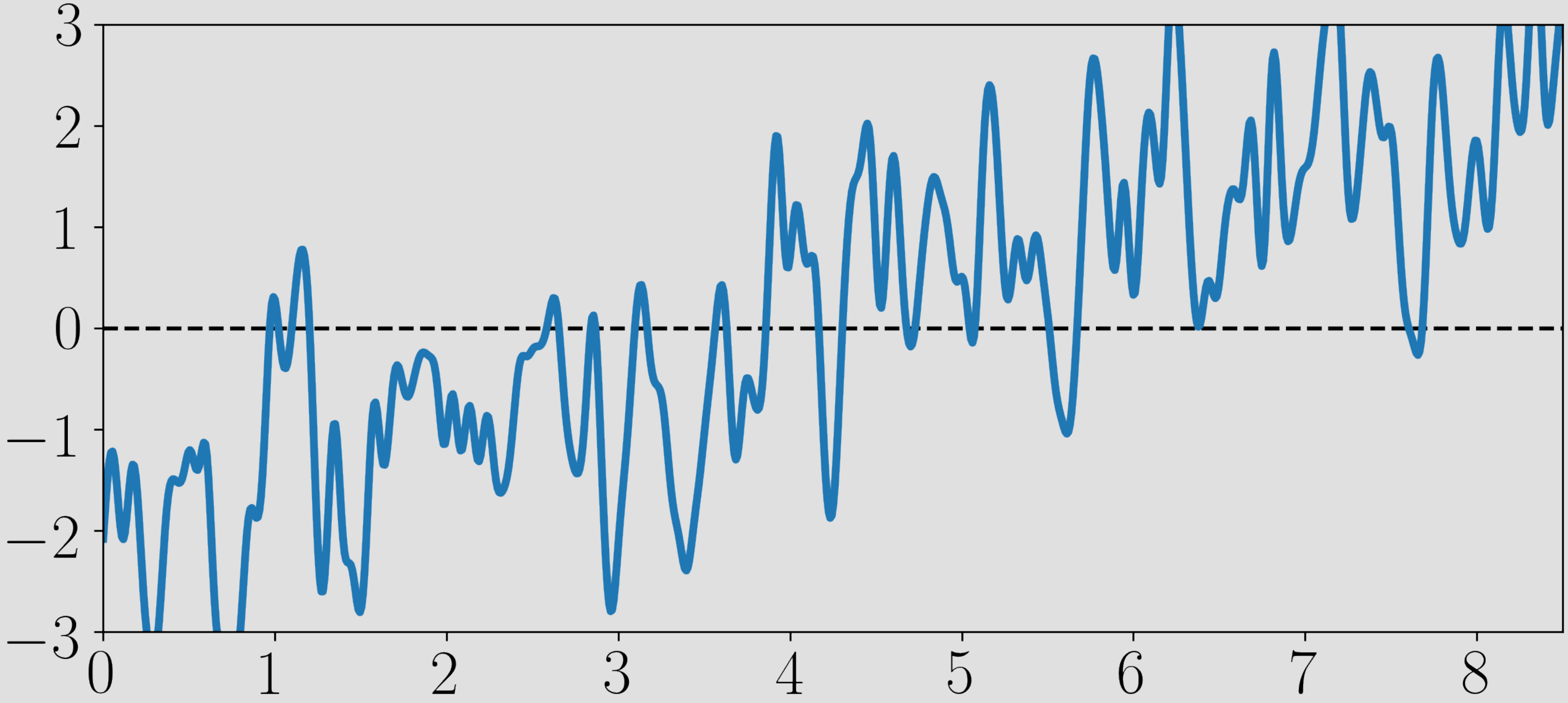
Stationary process

Stationary or not?



Stationary process

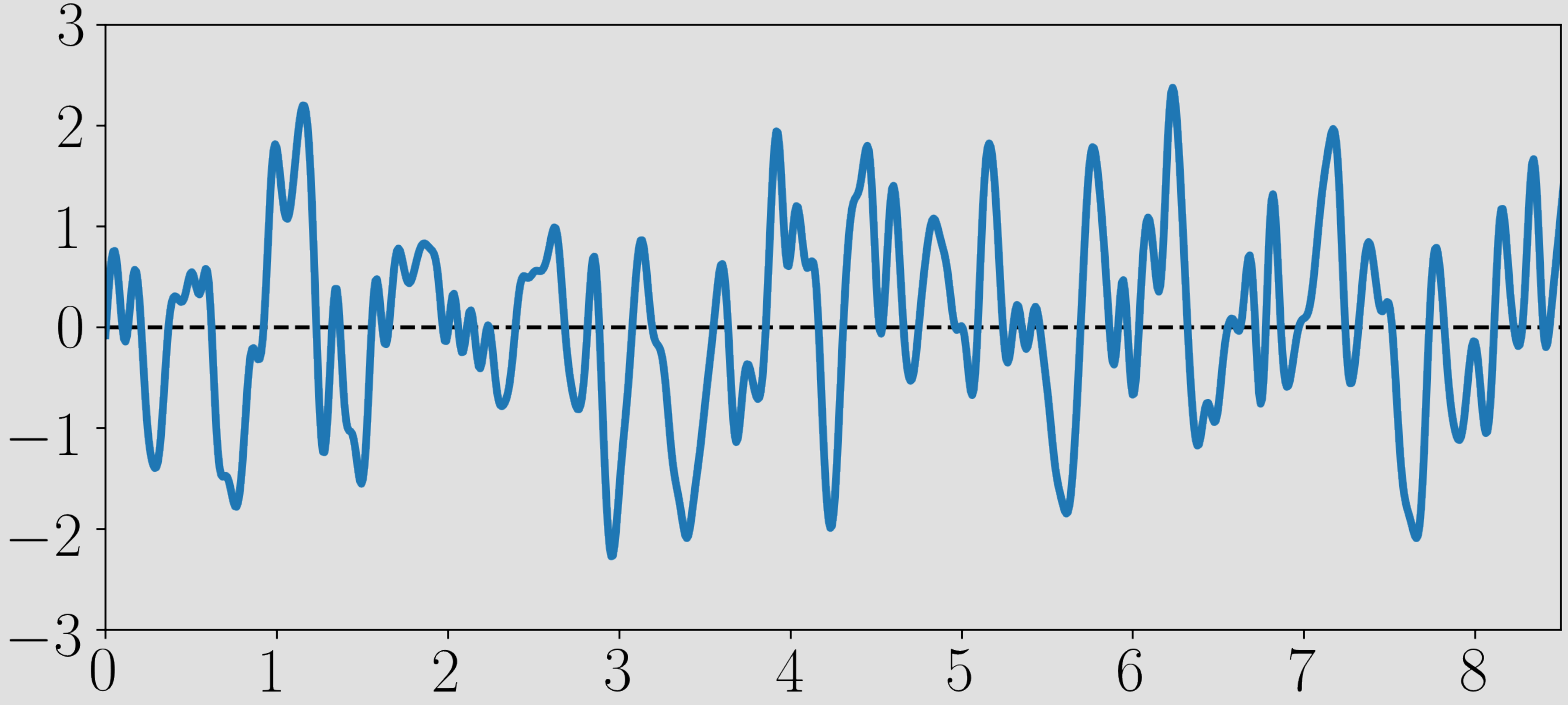
Stationary or not?



Not stationary: **trend**

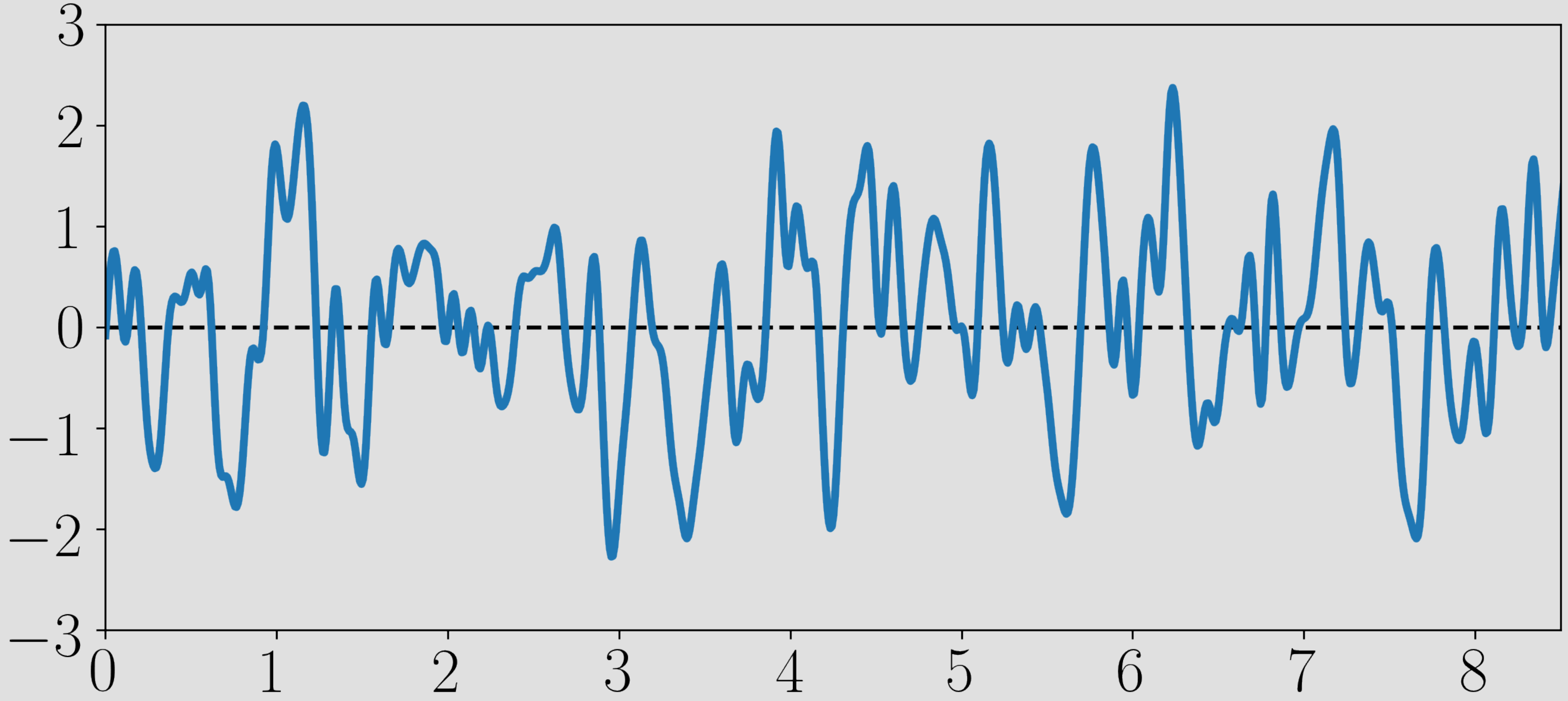
Stationary process

Stationary or not?



Stationary process

Stationary or not?



Stationary

Stationary process

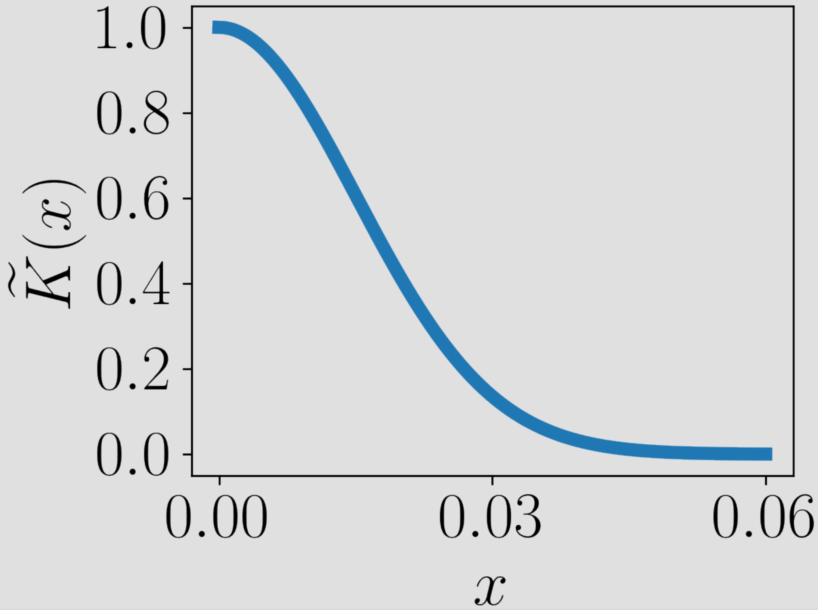
Definition:

Gaussian process is stationary, if:

$$m(x) = \text{const}$$

$$K(x_1, x_2) = \widetilde{K}(x_1 - x_2)$$

Variance: $\text{Var}[f(x)] = \widetilde{K}(0)$



Kernel

Radial Basis Function (RBF)

$$\widetilde{K}(x_1 - x_2) = \sigma^2 \exp\left(-\frac{(x_1 - x_2)^2}{2l^2}\right)$$

↳ length-scale

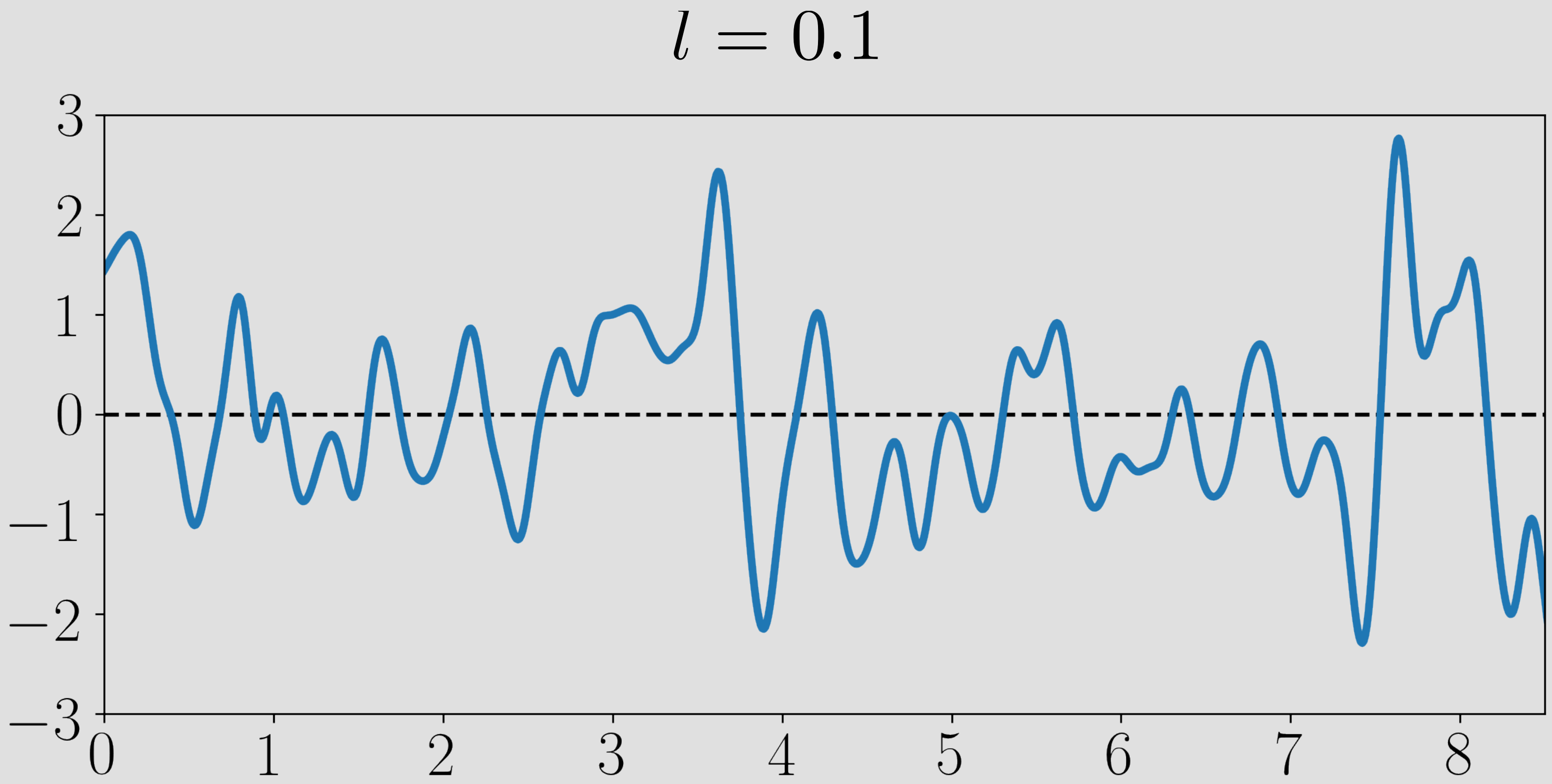
Rational Quadratic

$$\widetilde{K}(x_1 - x_2) = \sigma^2 \left(1 + \frac{(x_1 - x_2)^2}{2\alpha l^2}\right)^{-\alpha}$$

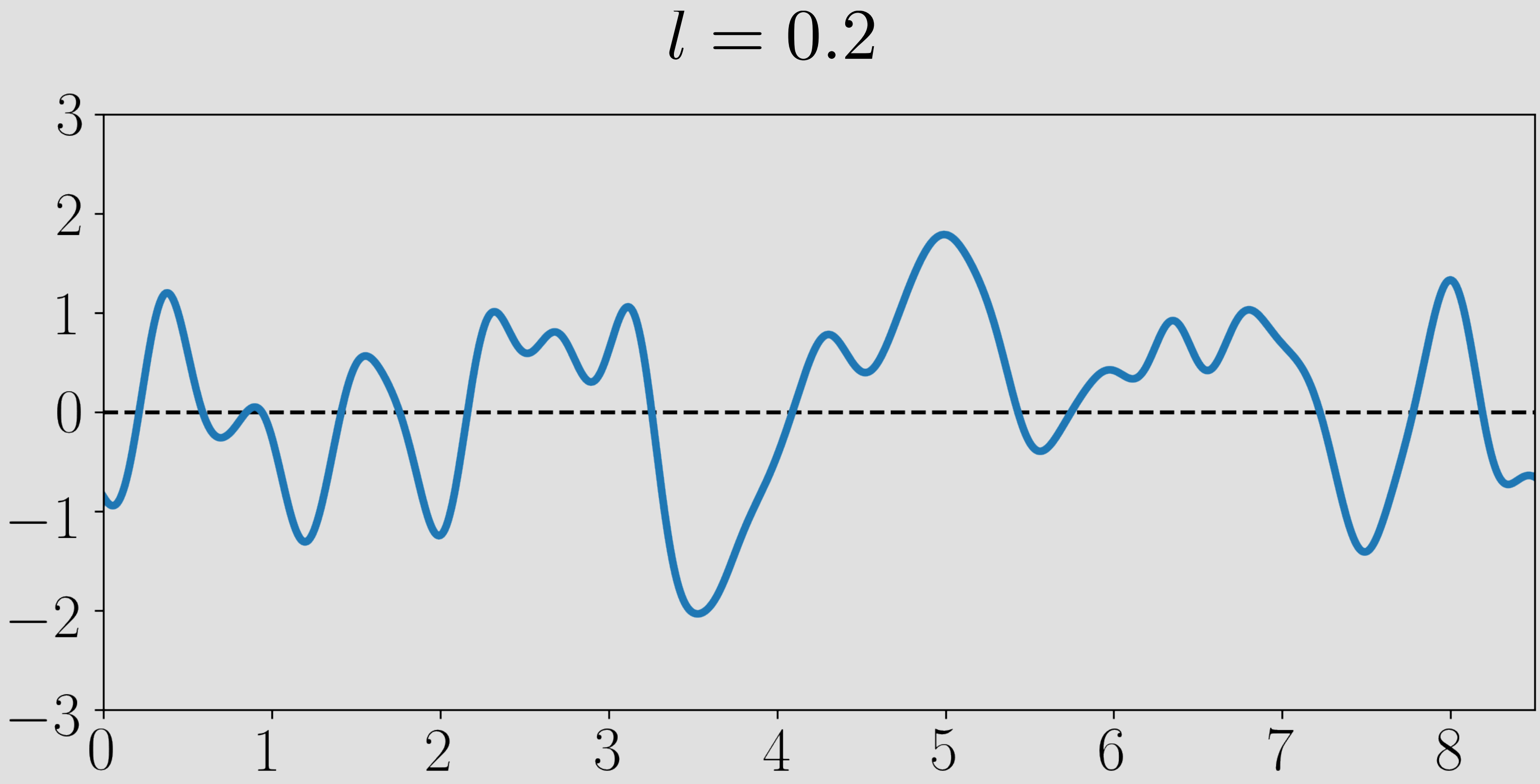
White noise

$$\widetilde{K}(x_1 - x_2) = \sigma^2 \delta(x_1 - x_2)$$

Kernel



Kernel



Kernel