Discrete Optimization

Constraint-based Scheduling

Goals of the Lecture

- Scheduling with Constraint Programming
  - modeling
  - global constraints
  - and some nice techniques ...

Motivation

- Very successful application area for constraint programming

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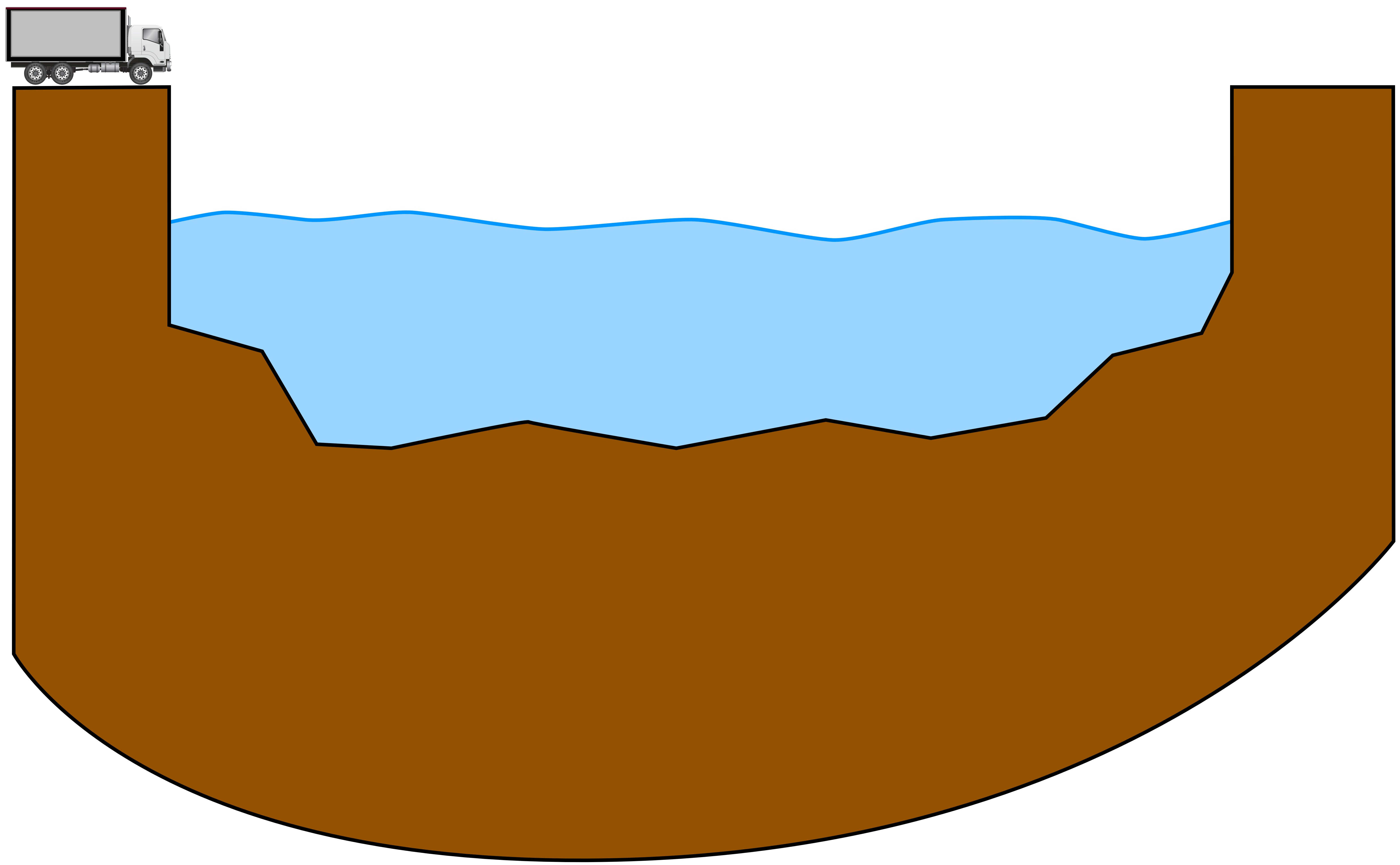
- minimize project duration subject to
  - precedence constraints
  - disjunctive constraints: no two tasks scheduled on the same machine can overlap in time

Project Scheduling



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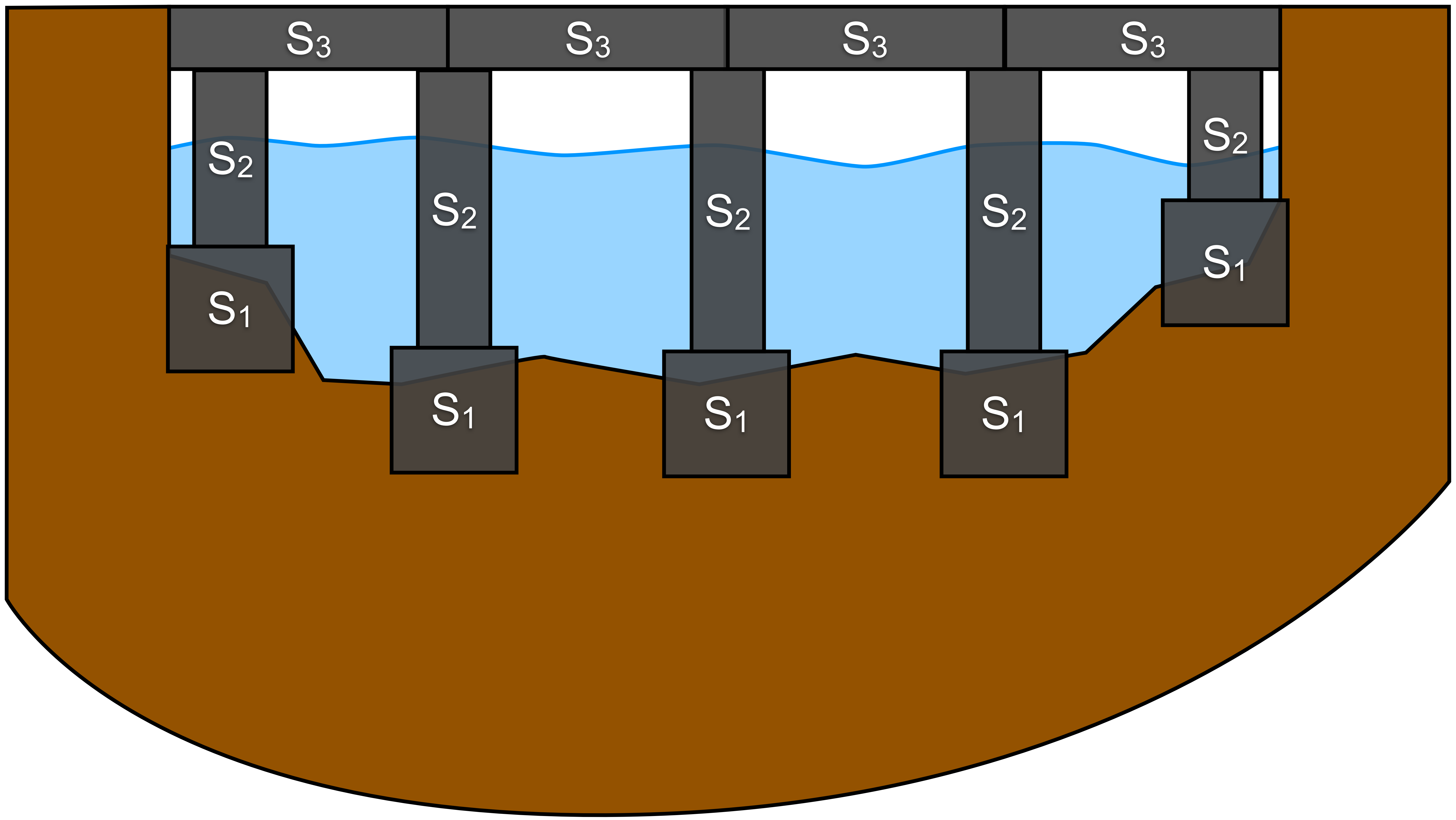

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- Dedicated abstractions for scheduling
  - model-based computing

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- Encapsulate
  - variables and global constraints
- Support
  - search procedures

Jobshop Scheduling

- The "TSP" of scheduling
  - standard benchmarks and open problems

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- Problem formulation
  - a set of tasks and
  - each task t has a duration d(t)
  - each task t executes on a machine m(t) and no two tasks scheduled on the same machine can overlap in time
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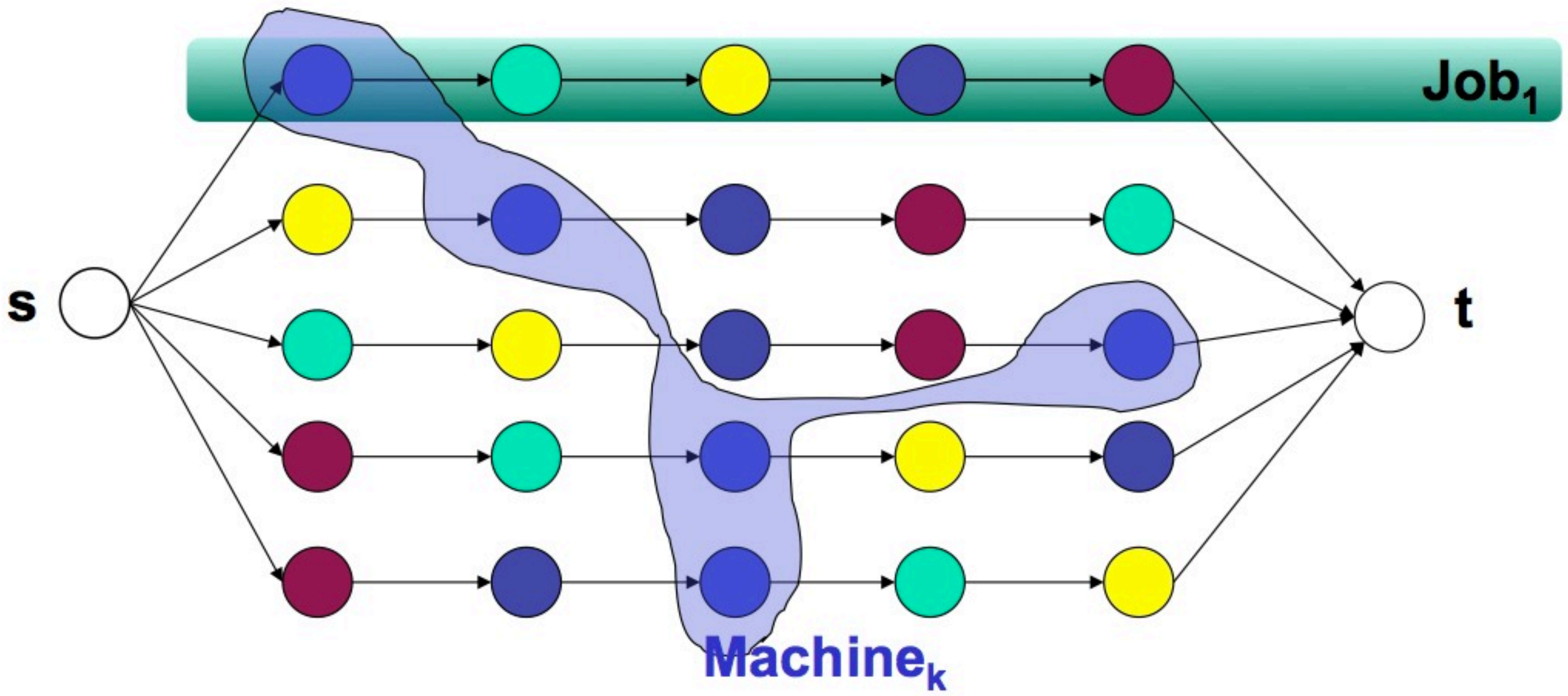
- Objective
  - minimize the project completion time

Jobshop Scheduling



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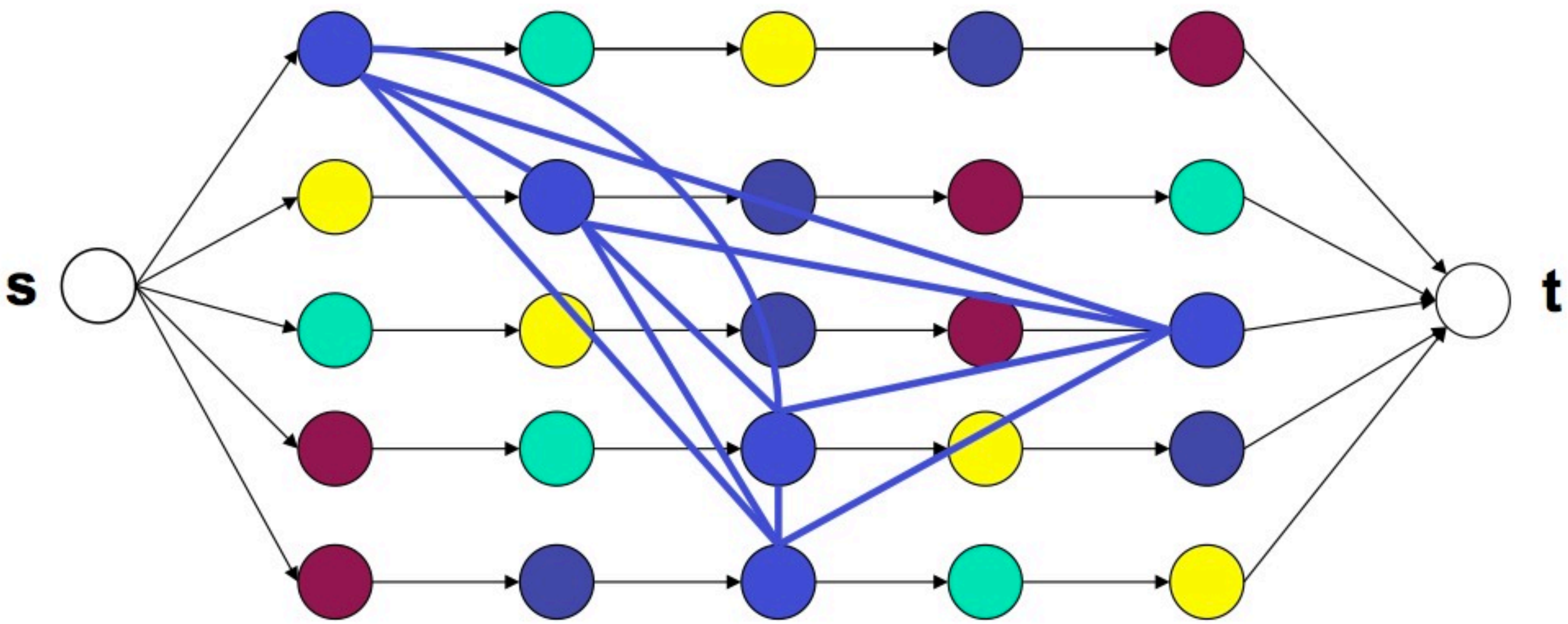

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  - a solution must find an ordering of the tasks on each machine

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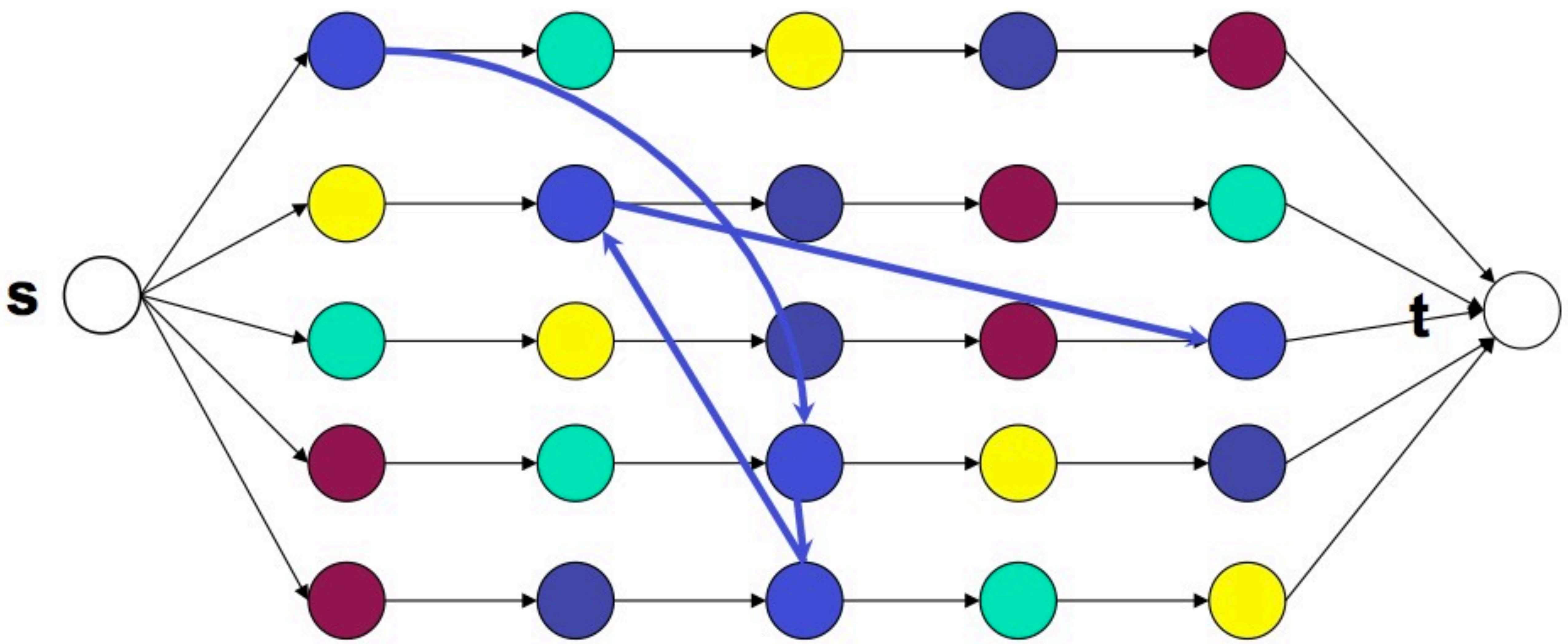

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Jobshop Scheduling

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Jobshop Scheduling

- It is sufficient to order all machines
  - each ordering then introduces precedence constraints
- Minimize project duration under precedence constraints
  - polynomial time
  - topological sorting (PERT)
  - transitive closure (Floyd-Warshall)

Jobshop Scheduling

```
int duration[Jobs,Tasks] = ...;
int machine[Jobs,Tasks] = ...;
int horizon = sum(j in Jobs,t in tasks) duration[j,t];

Scheduler sched(horizon);
Activity act[j in Jobs,t in Tasks](sched,duration[j,t]);
Activity makespan(sched,0);
UnaryResource r[Machines](sched);

minimize makespan.end
subject to {
   forall(j in Jobs,t in tasks: t != Tasks.high)
       act[j,t] precedes act[j,t+1];
   forall(j in Jobs)
       act[j,Tasks.high] precedes makespan;
   forall(j in Jobs,t in Tasks)
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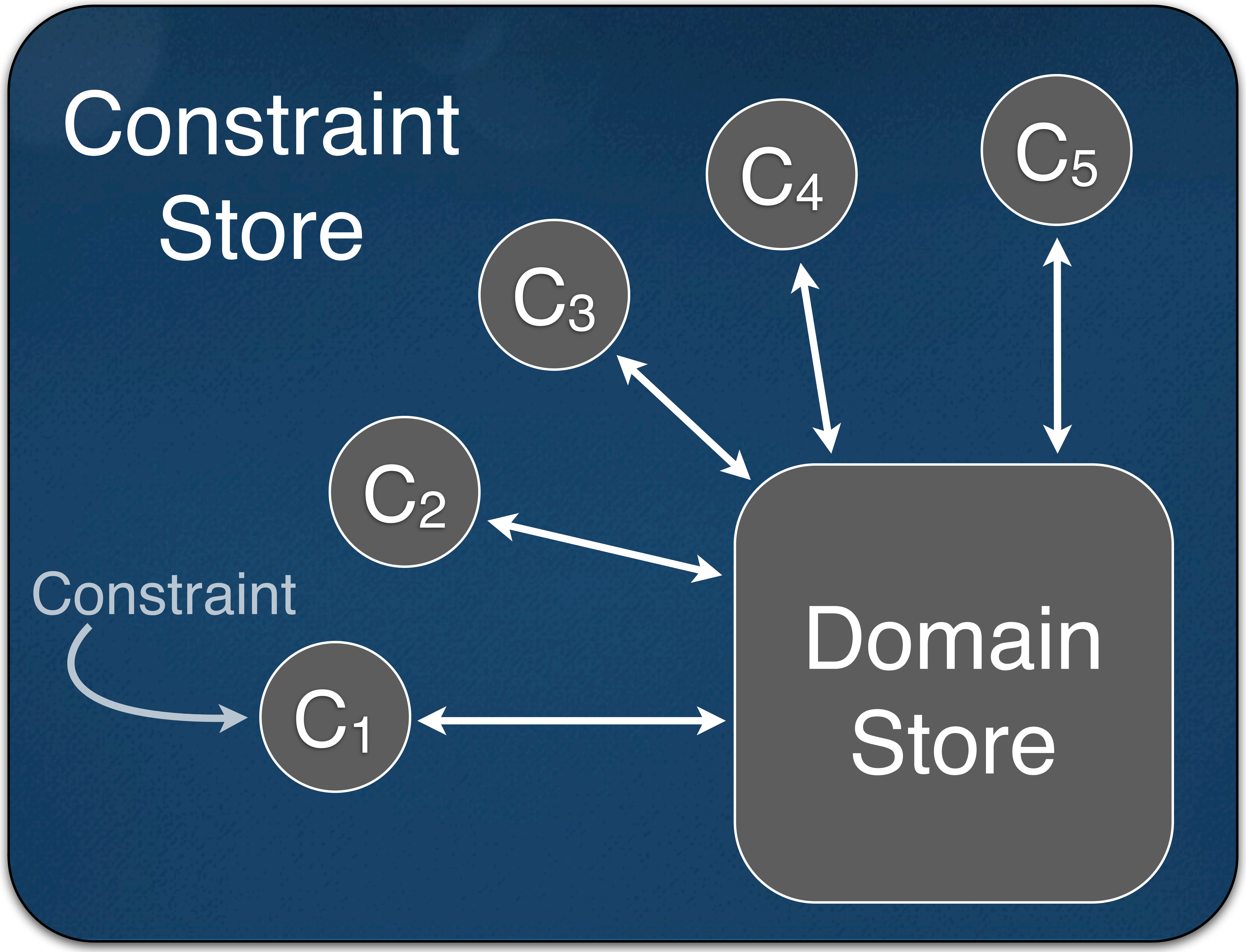
Model Compilation

- Each activity encapsulates
  - variables (e,s,d) for starting date, ending date, and duration
  - a constraint linking these three variables
    - $s + d = e$
- Each precedence constraint (b,a)
  - $s_a \geq e_b$
- Each machine m gives rise to a global constraint
  - disjunctive($t_1,\ldots,t_n$)

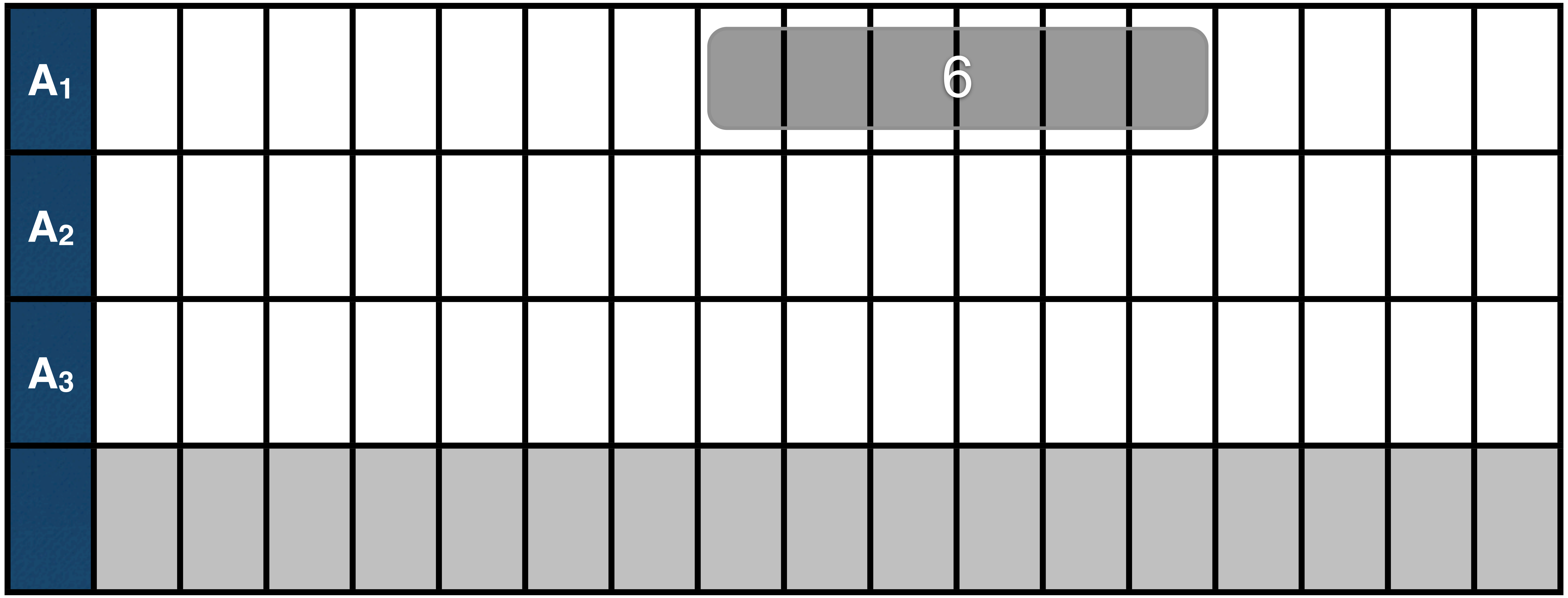
Jobshop Scheduling



Disjunctive Constraint: Feasibility

A_1																	
A_2																	
A_3																	

Disjunctive Constraint: Feasibility



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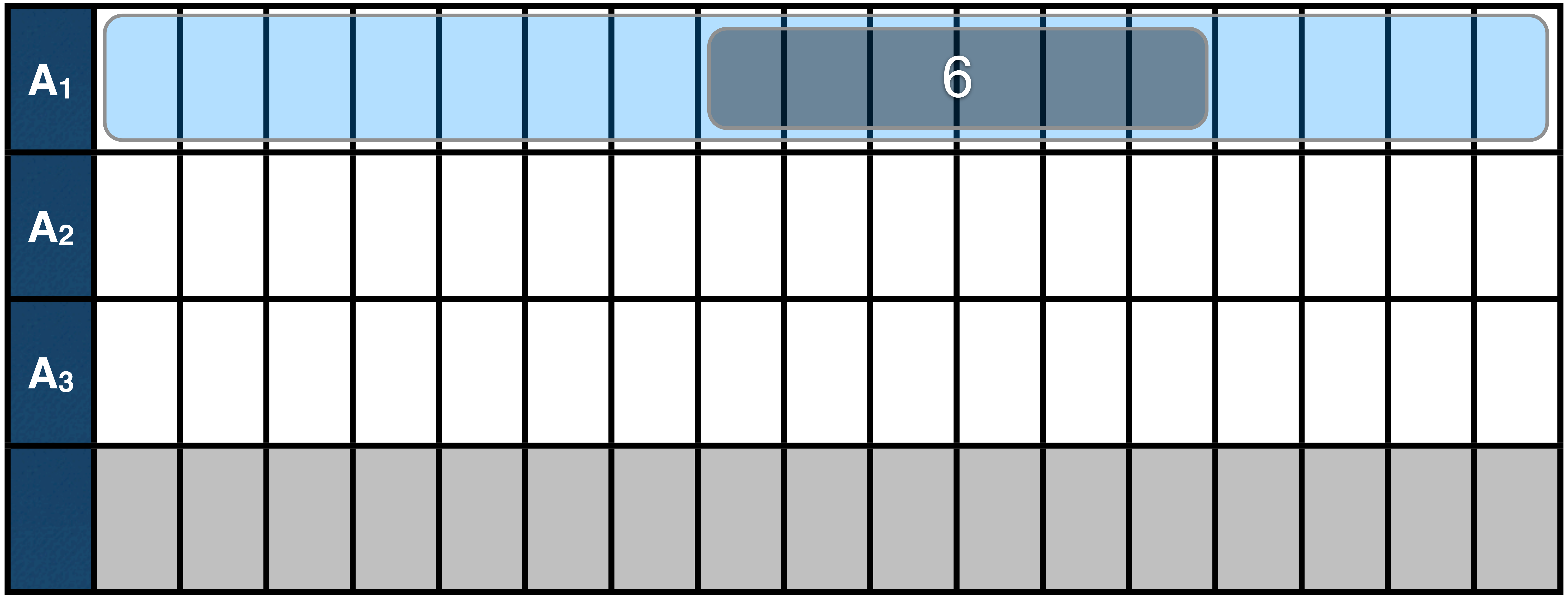

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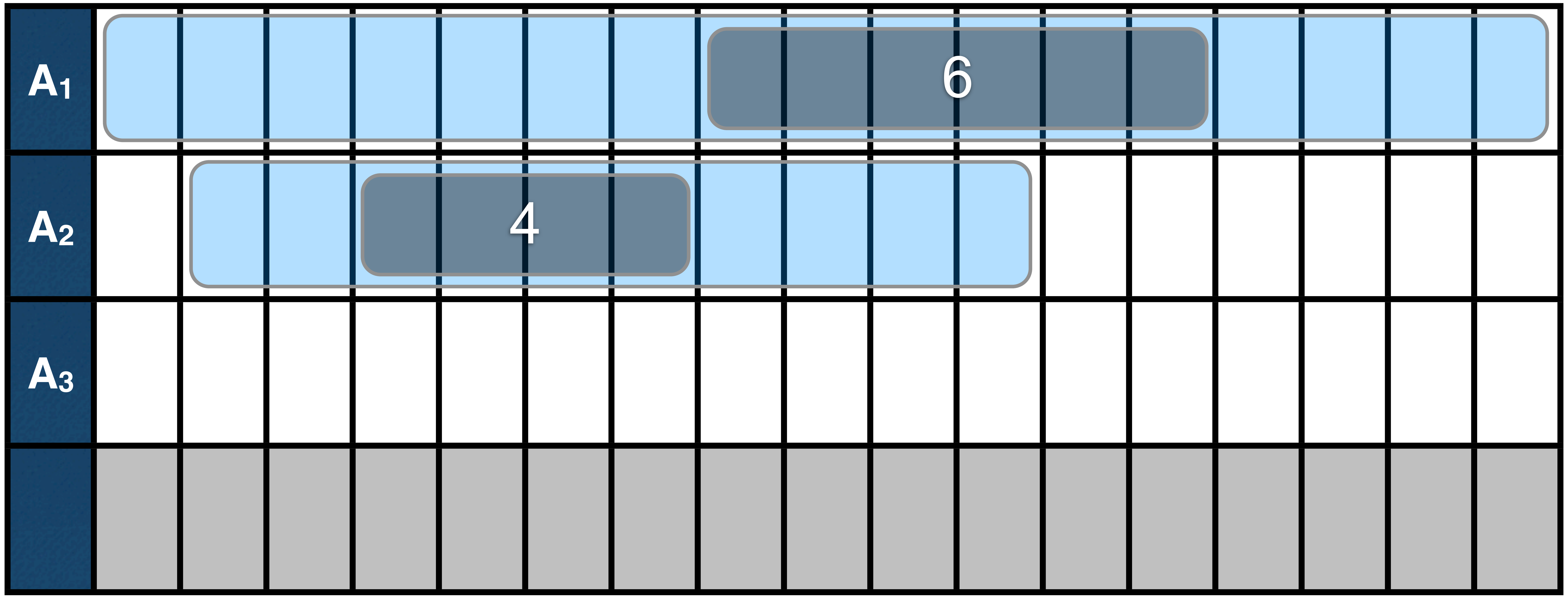

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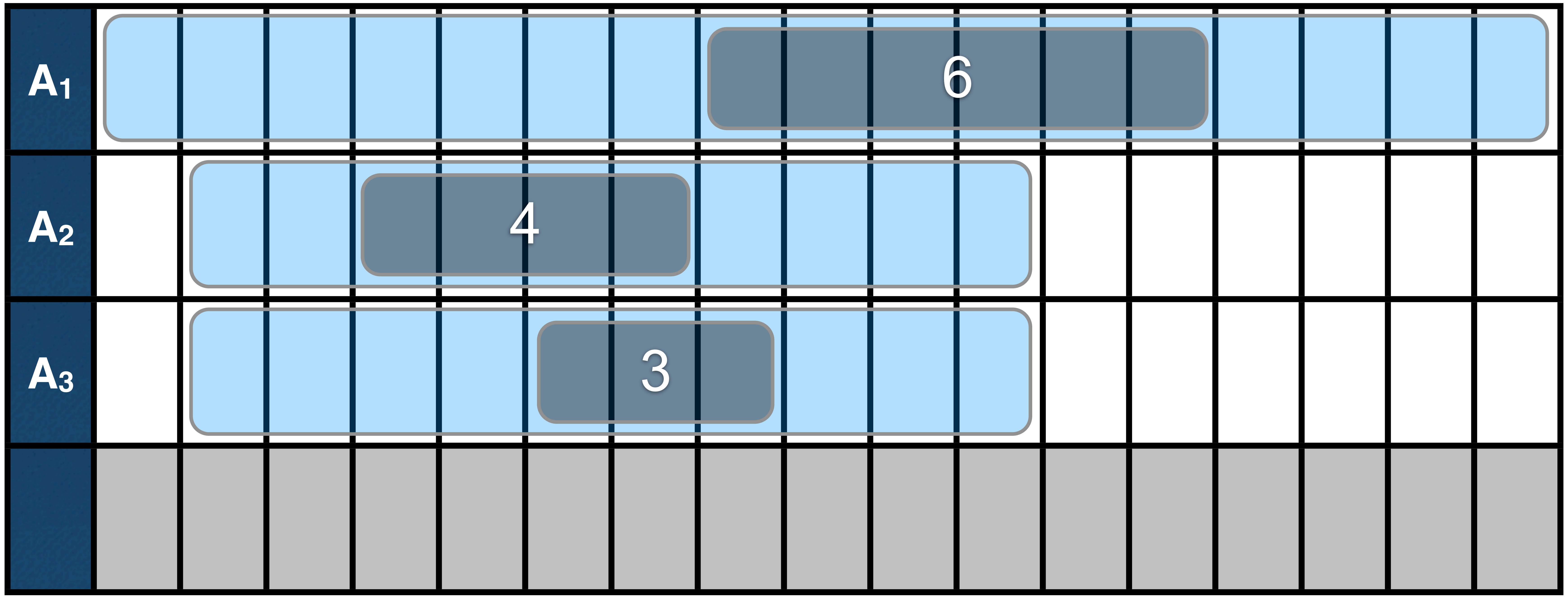

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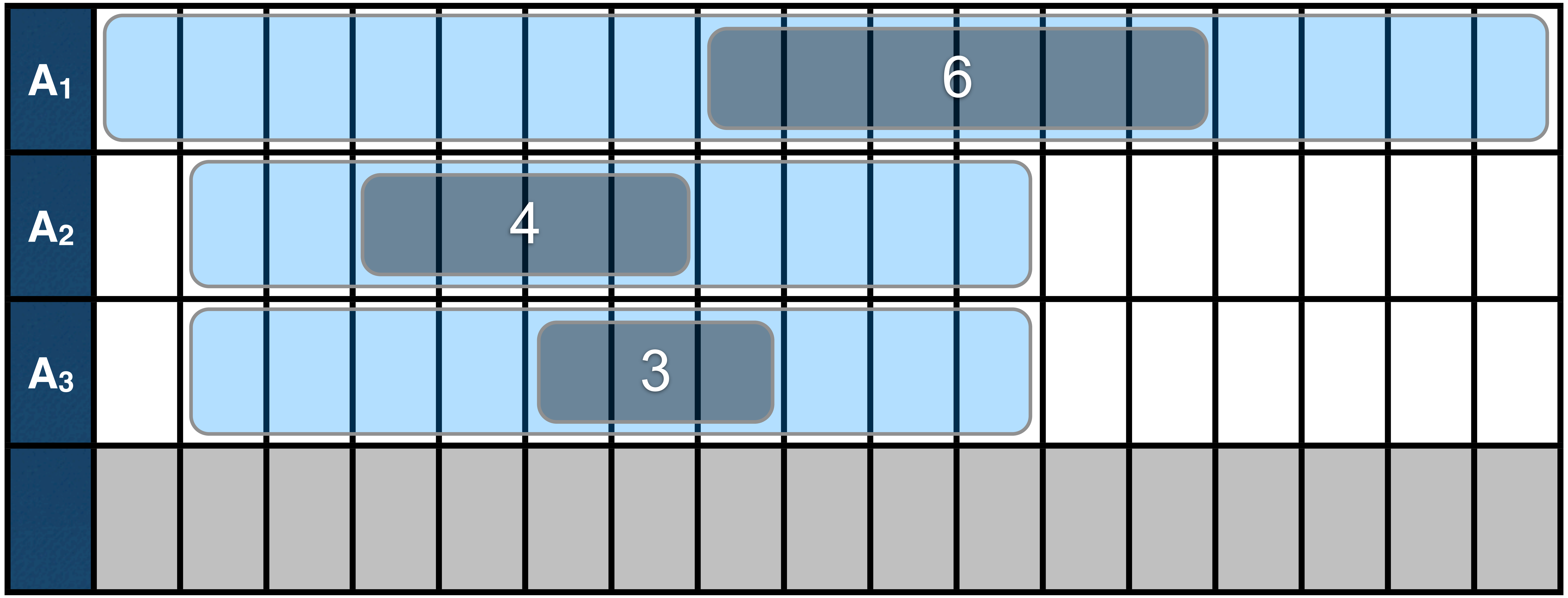

- Detecting feasibility of a disjunctive constraint is NP-Complete

Feasibility of Disjunctive Constraints

- Some basic intuition and algorithms
  - very rich domain
  - just making you curious
  - many interesting connections
- Notations
  - $s(\Omega)$ = min(t in Ω) min(s_t)
  - $e(\Omega)$ = max(t in Ω) max(e_t)
  - $d(\Omega)$ = sum(t in Ω) min(d_t)

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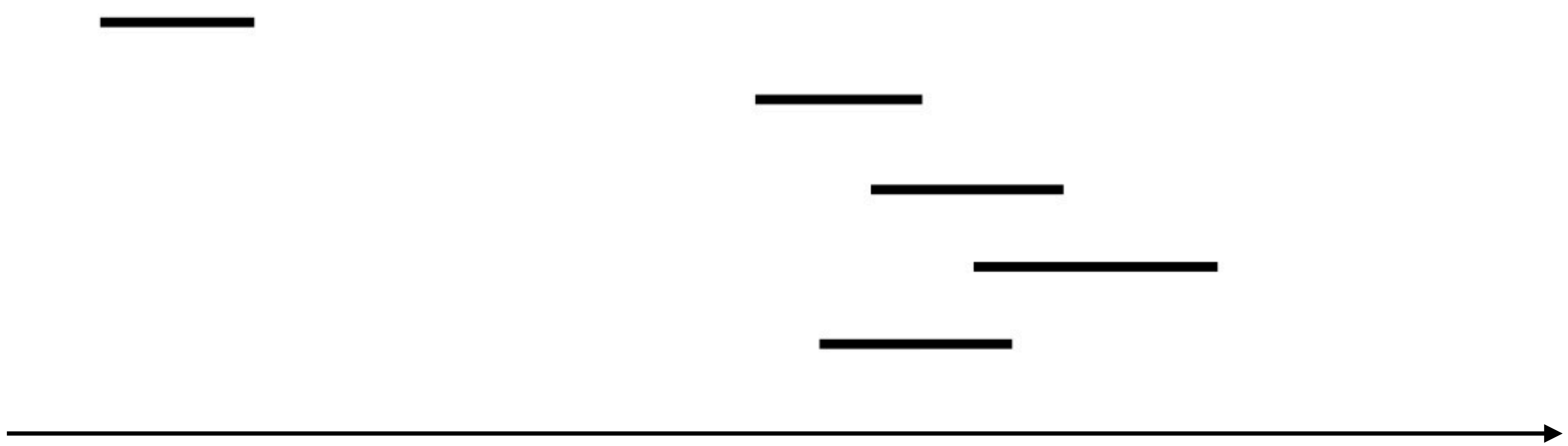

Disjunctive Feasibility

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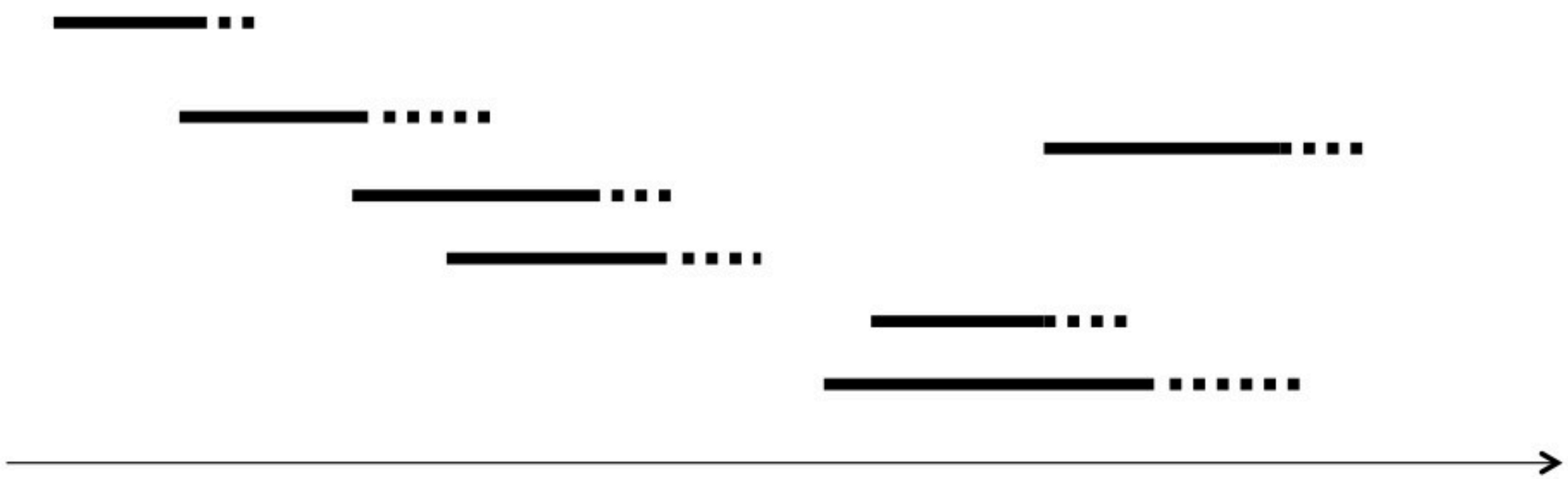

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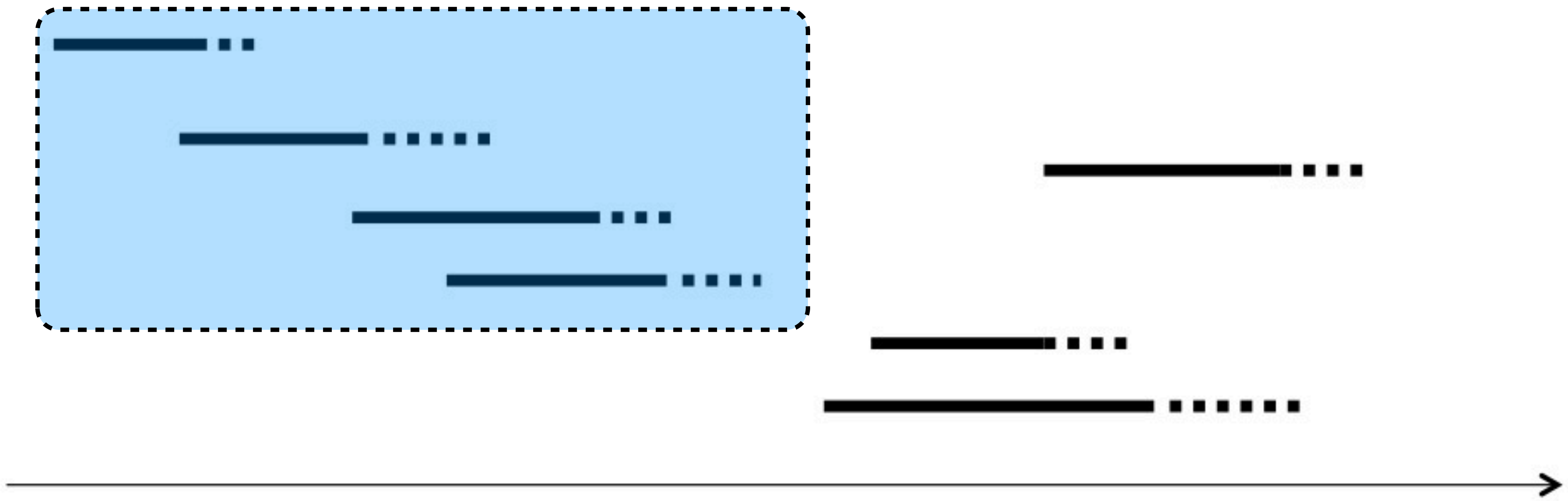

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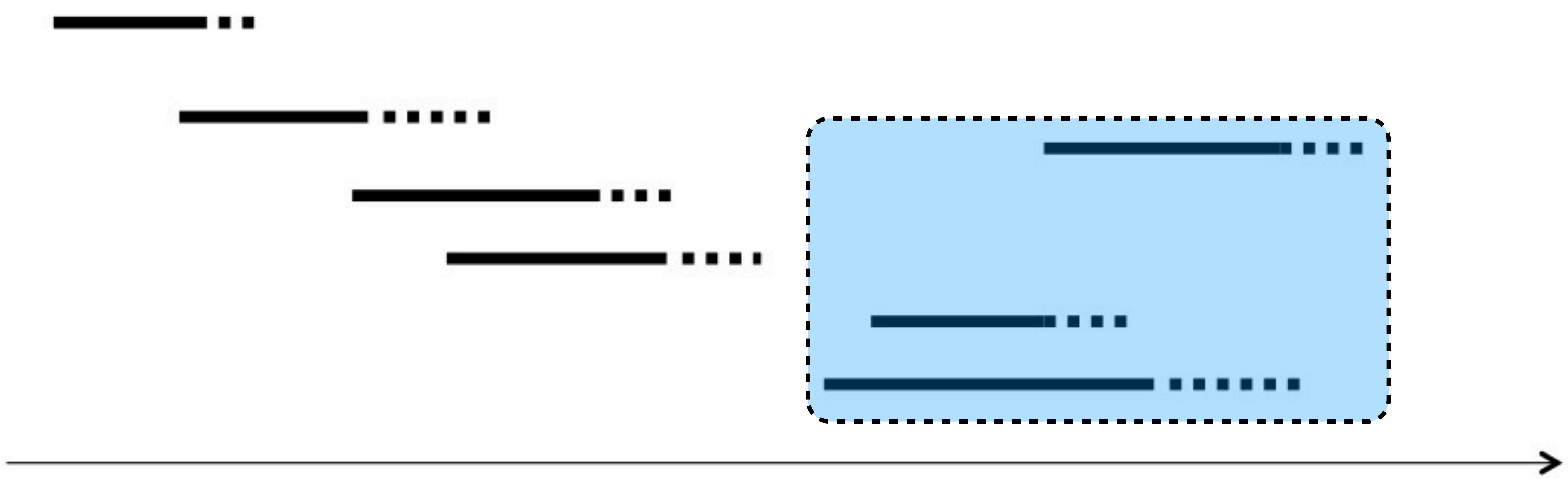

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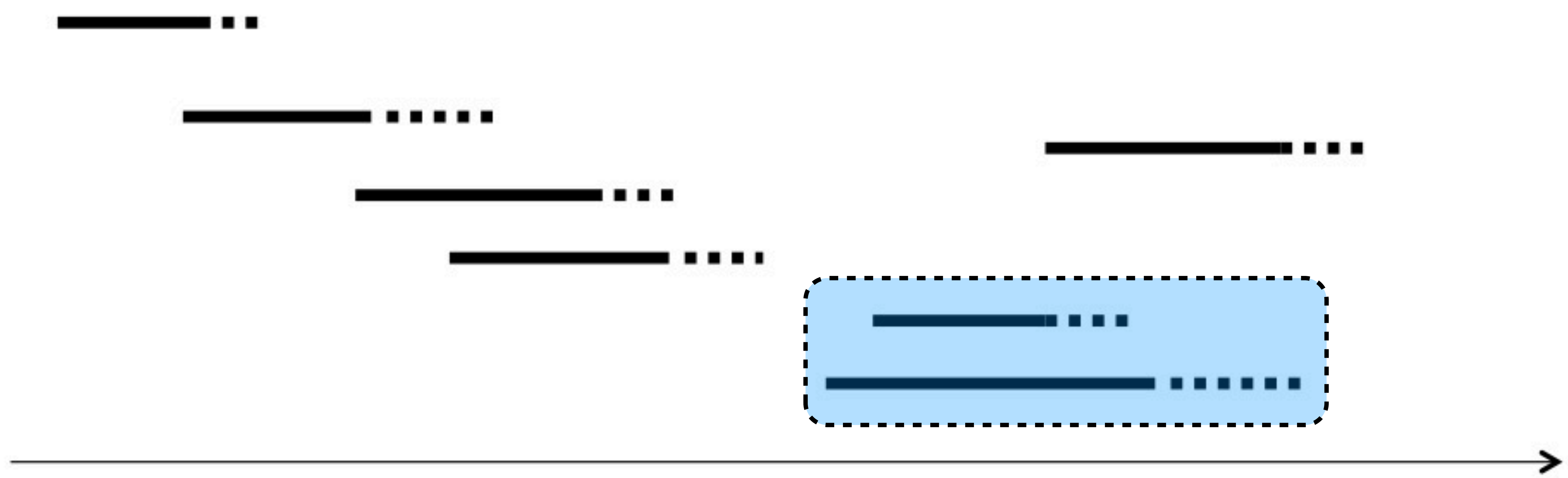

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- What is the issue here?

Disjunctive Feasibility

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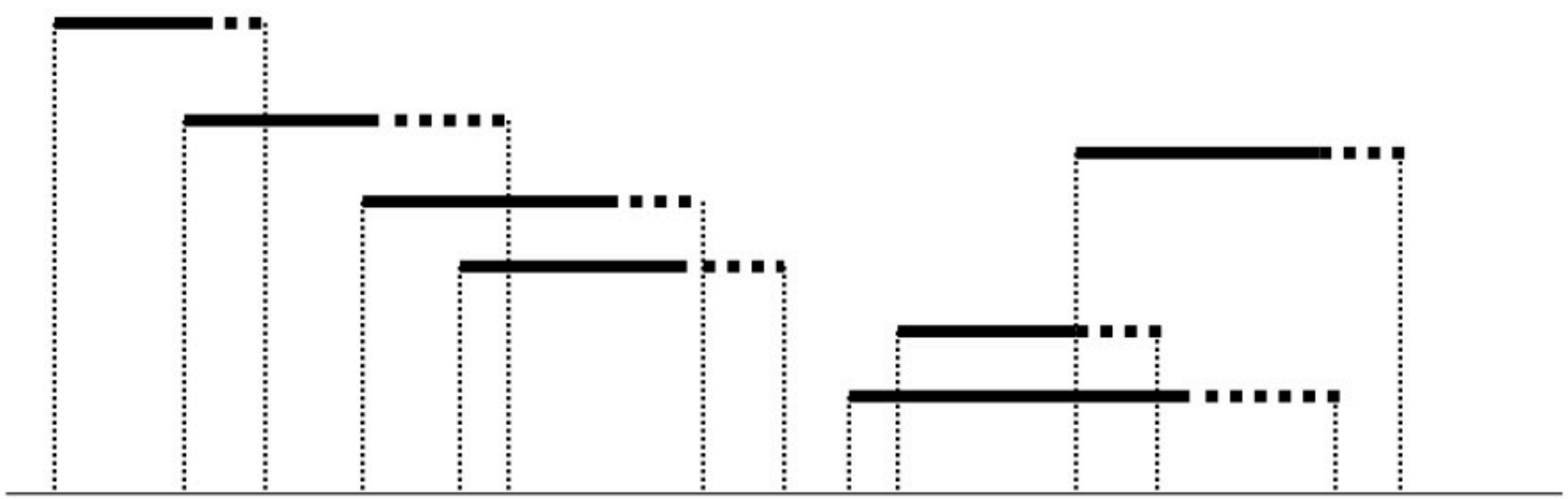

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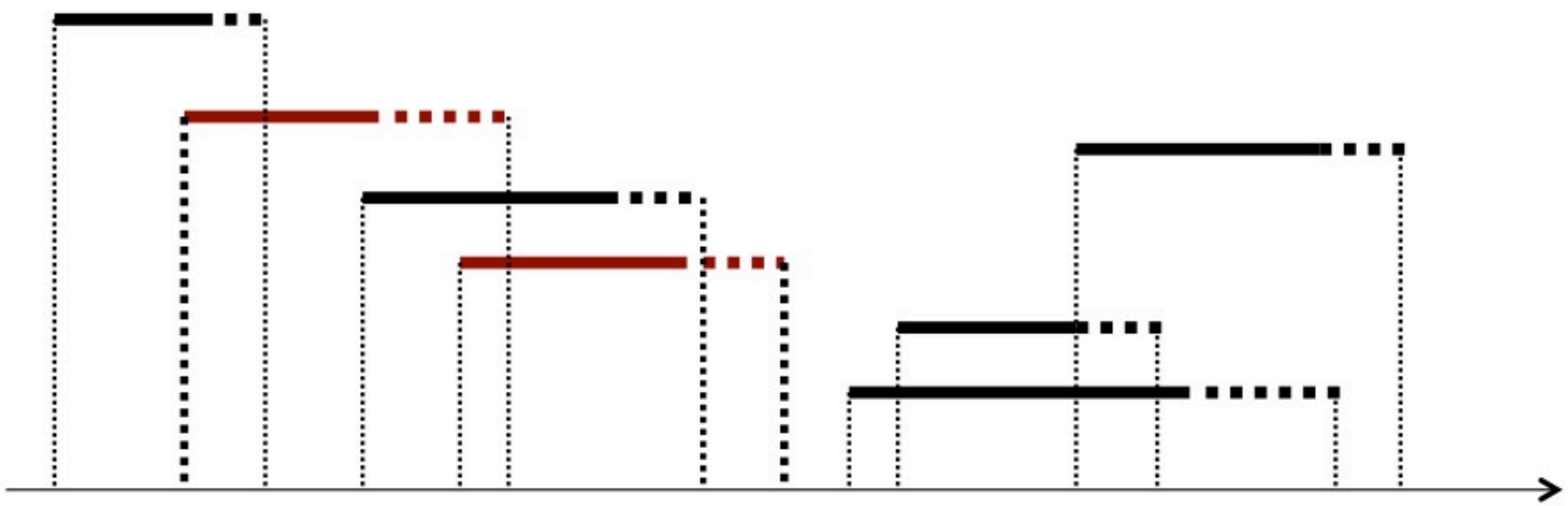

Disjunctive Feasibility

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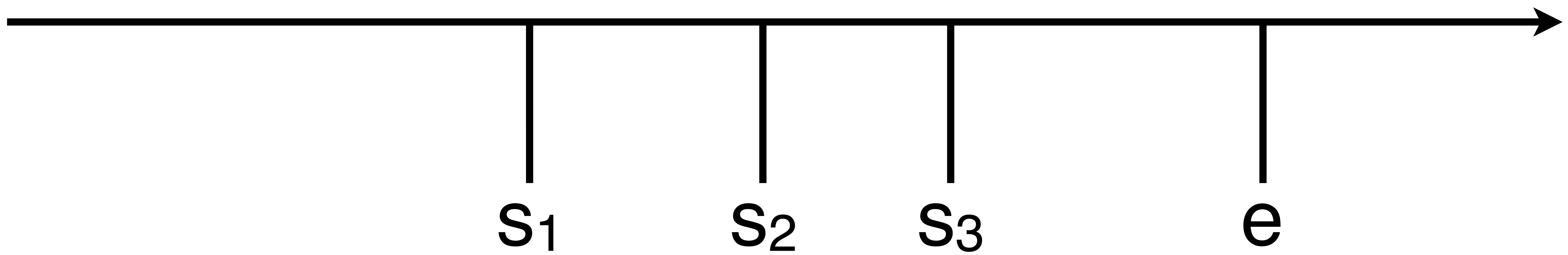
- Complexity
  - $O(|T|^3)$

Disjunctive Feasibility



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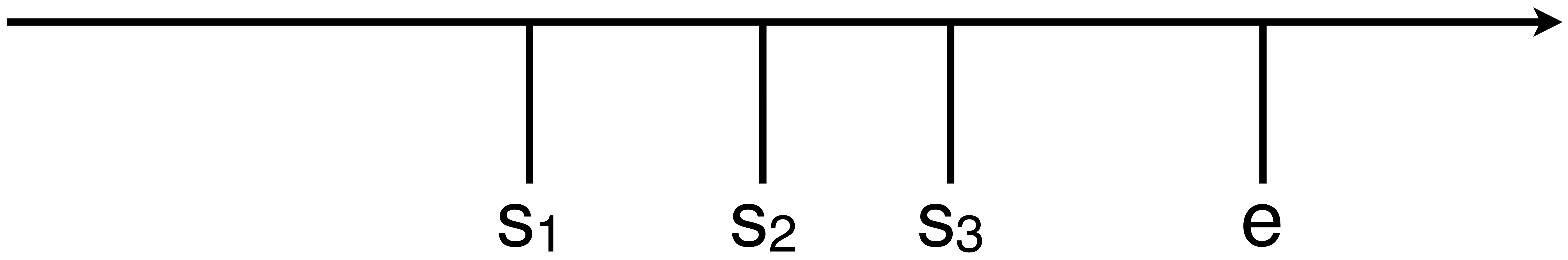
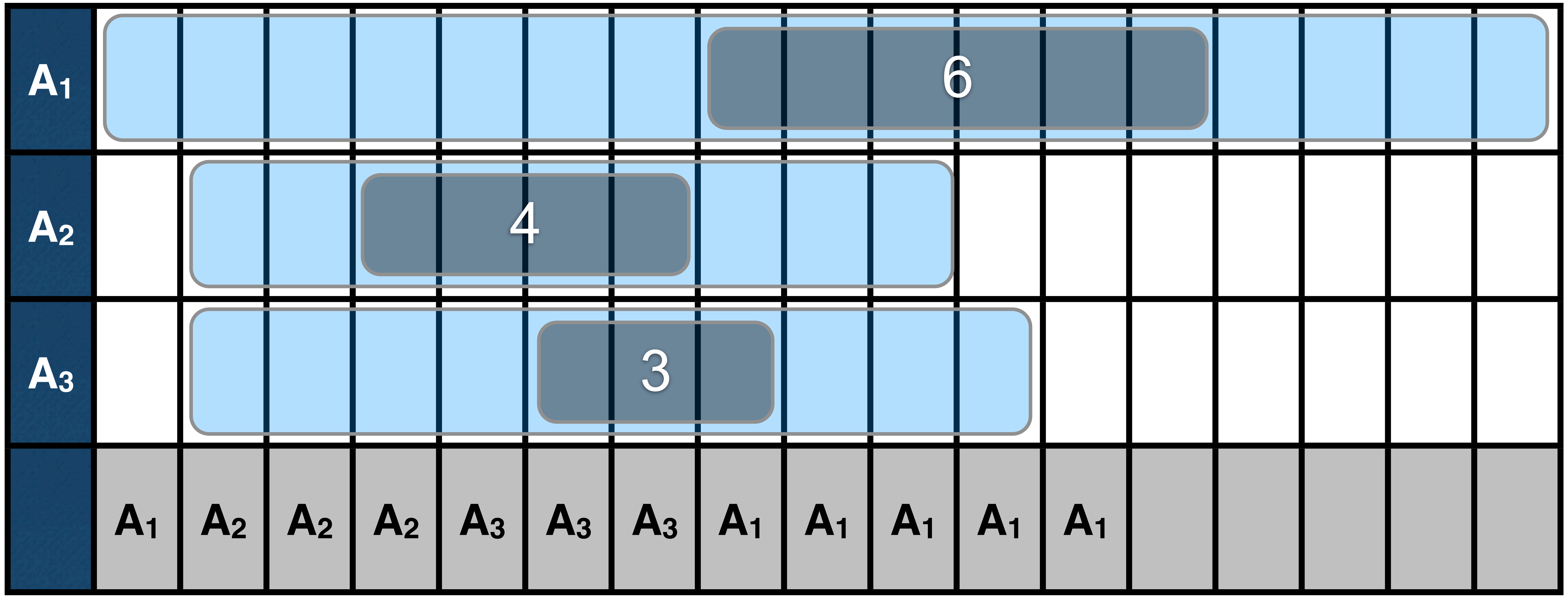



```
d := 0;
for each task t in decreasing order of st
   if et <= e
     d := d + dt;
     if st + d > e
       return failure;
return success;
```

Disjunctive Constraint: Feasibility

- Relax: feasibility of the preemptive schedule



- One-machine preemptive feasibility can be computed in O(|T| log |T|)

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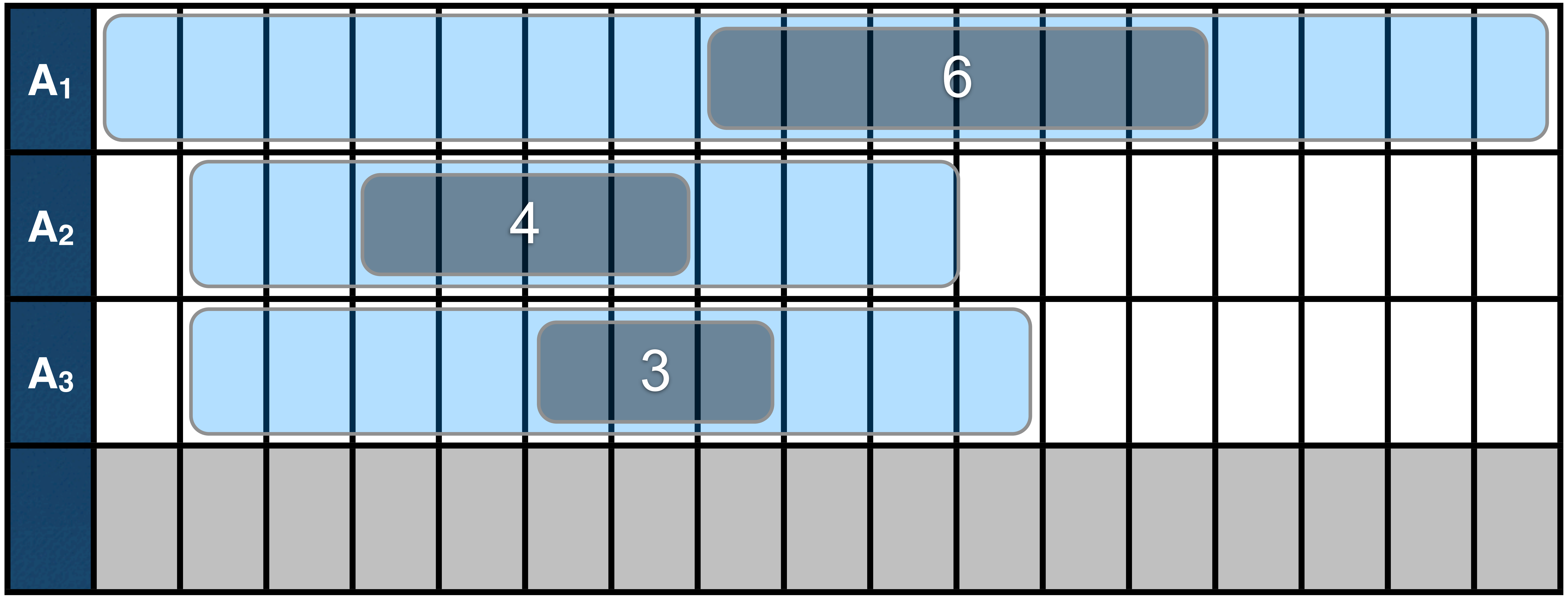
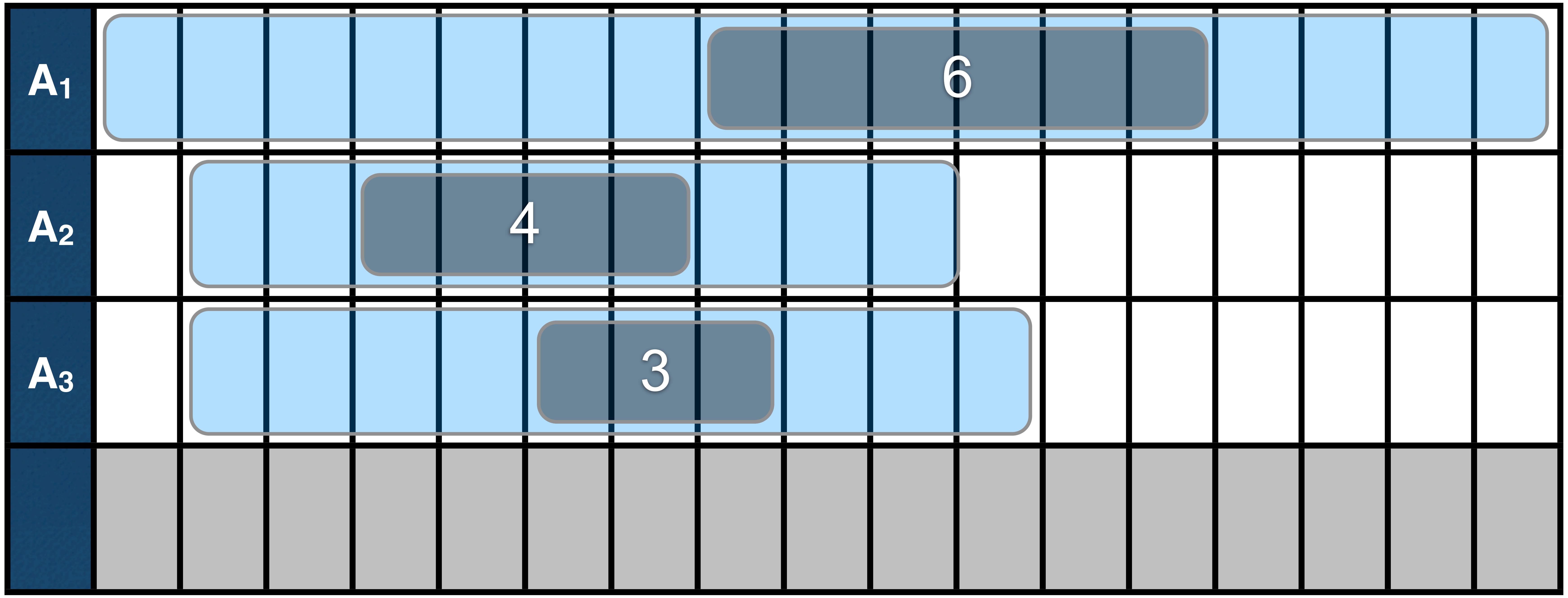
- The edge finding rules can be enforced in strongly polynomial time

Disjunctive Pruning



Disjunctive Pruning

- Can A_1 start before A_2 or A_3?



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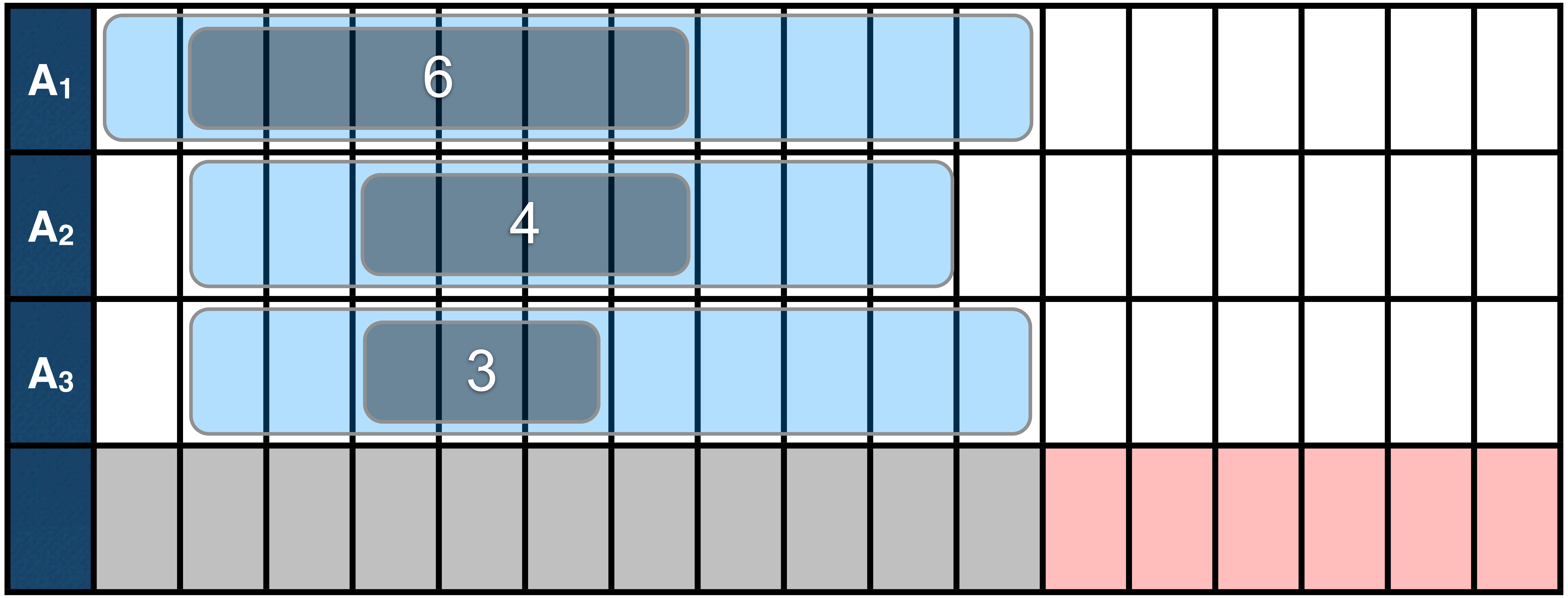
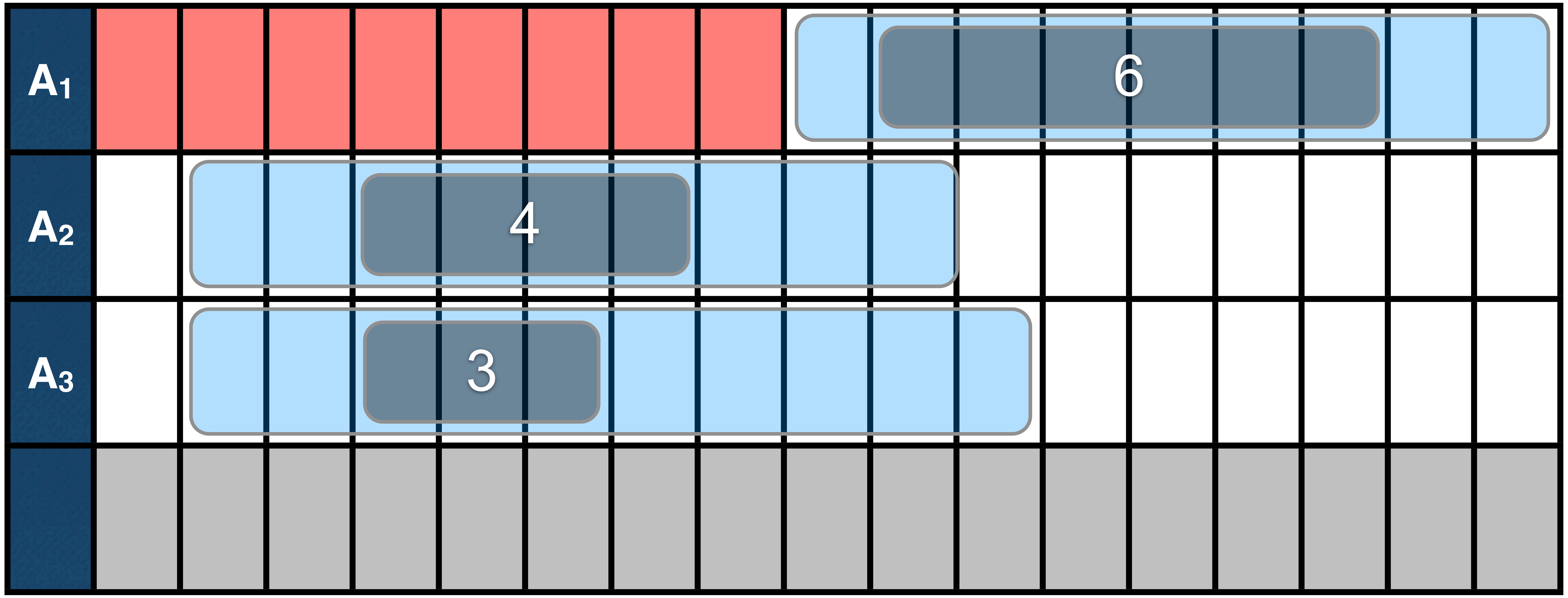


Disjunctive Pruning

- A1 must start after A2 and A3



Search for Disjunctive Scheduling

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- Which task?
  - a task that can be scheduled first (or last)
  - a task that is as tight as possible

Until Next Time