Discrete Optimization

Column Generation

Goals of the Lecture

- Column generation
  - introduction
  - cutting stock

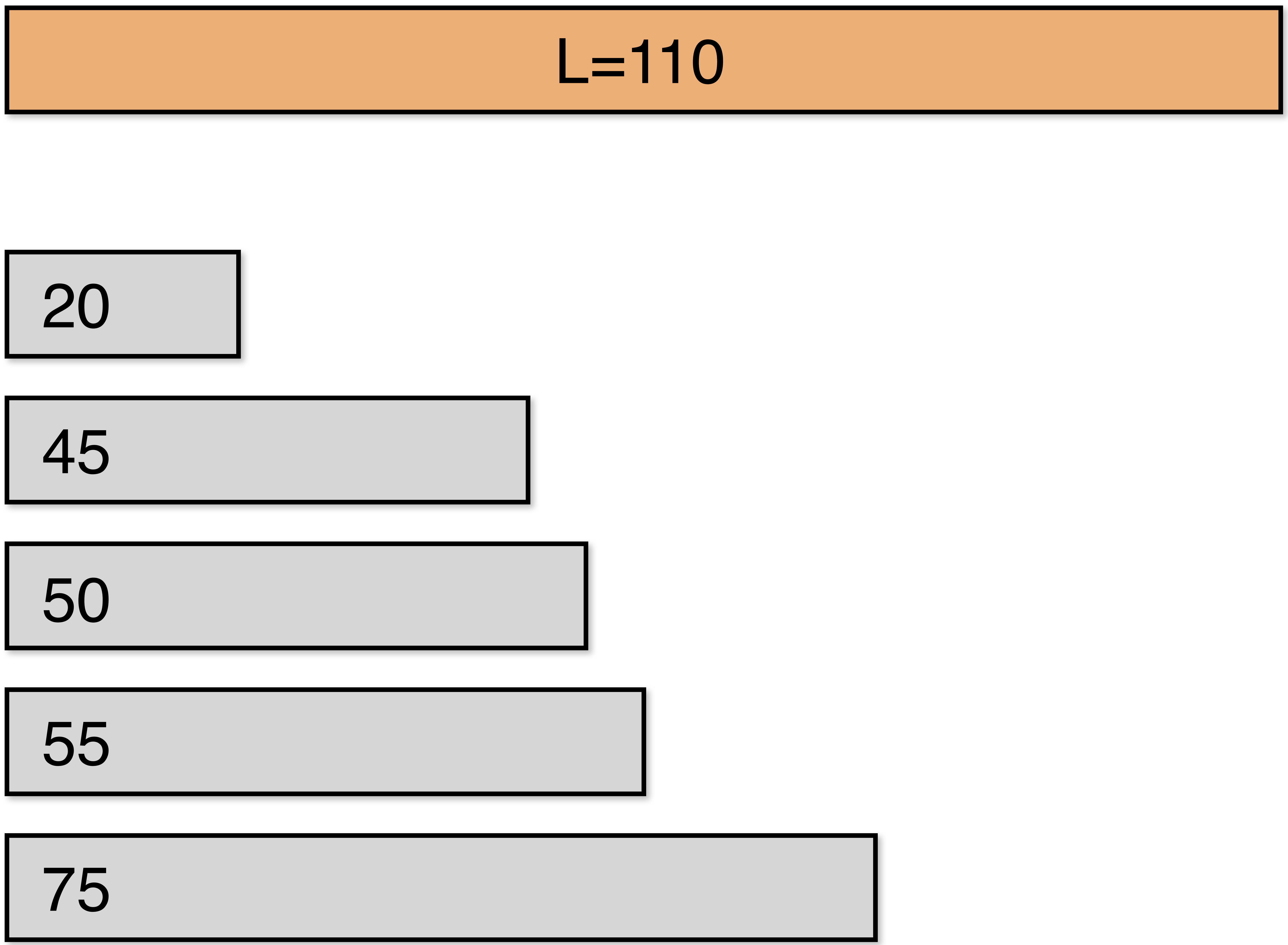
Motivation

- Branch and Cut
  - solving a MIP with exponentially many constraints
    - subtour constraints
  - generate the constraints on demand
- Column generation
  - solving a LP with exponentially many variables
  - variables represent complex objects
- Branch and Price
  - solving a MIP with exponentially many variables
  - branching over column generation

Cutting Stock (Gilmore and Gomory)

- Given
  - a number of large wood boards of a length L
  - a number of shelves of various sizes that need to be cut from the boards
  - the demand for each shelf size
    - how many to cut.
- Find
  - the smallest number of boards to cut in order to meet the demand for shelves

Cutting Stock (Gilmore and Gomory)



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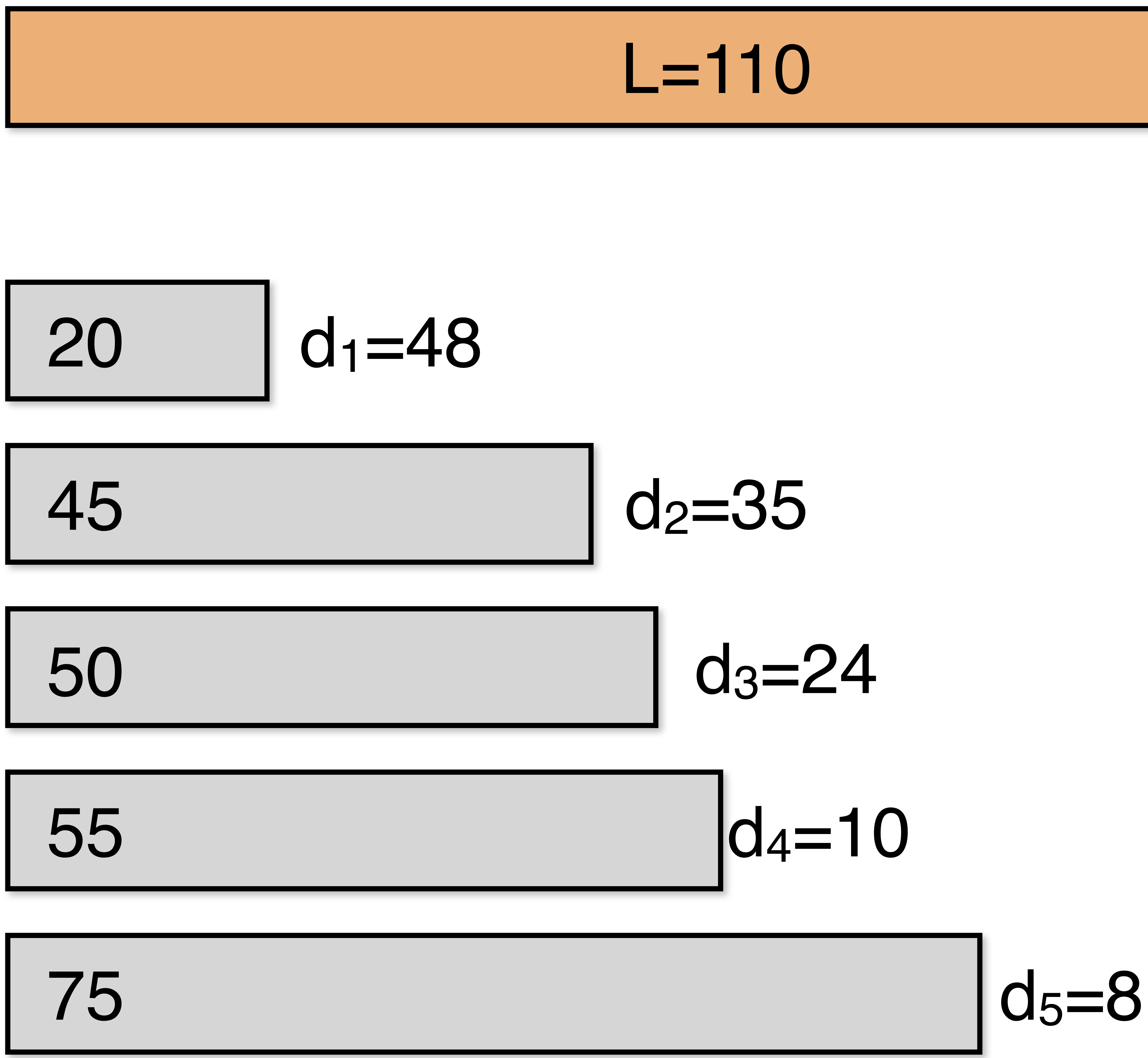

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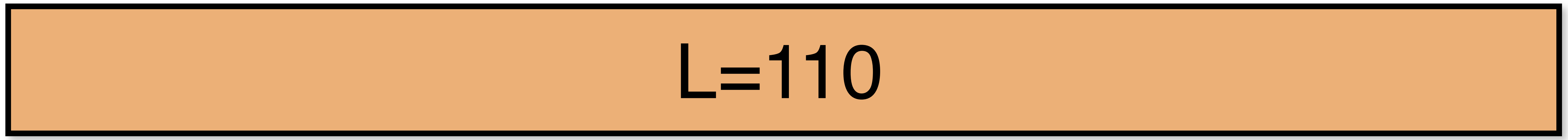

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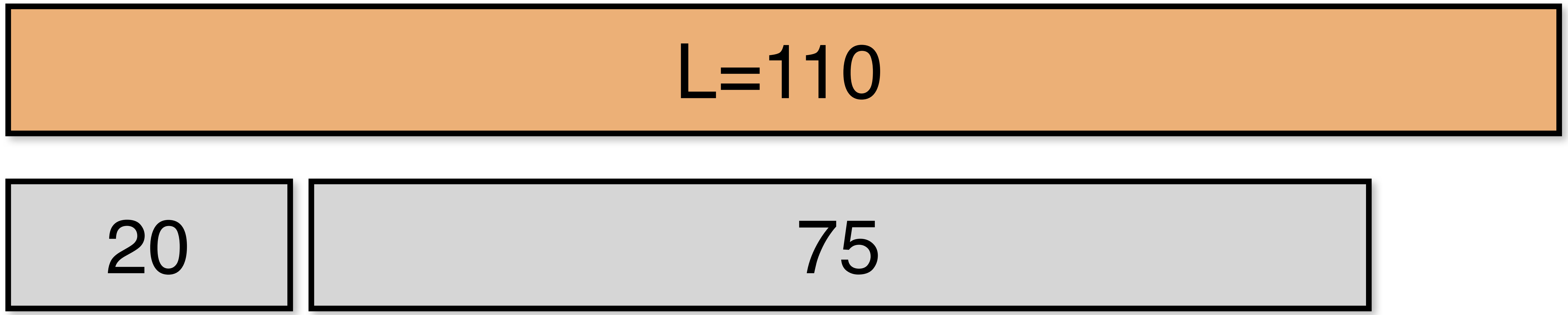

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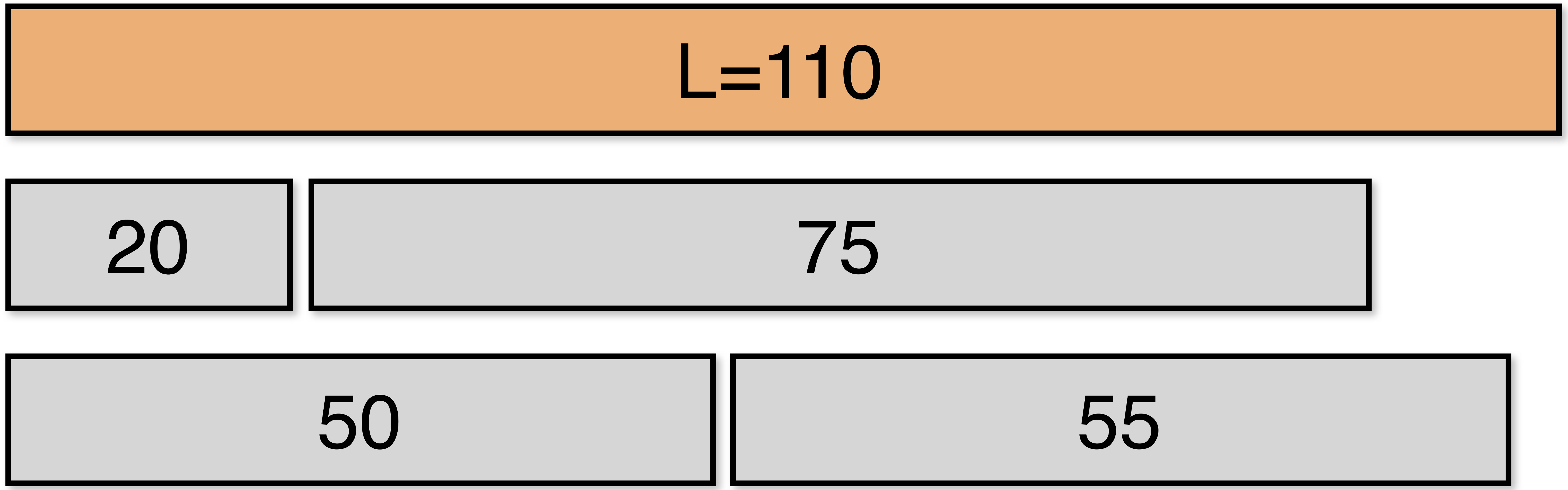

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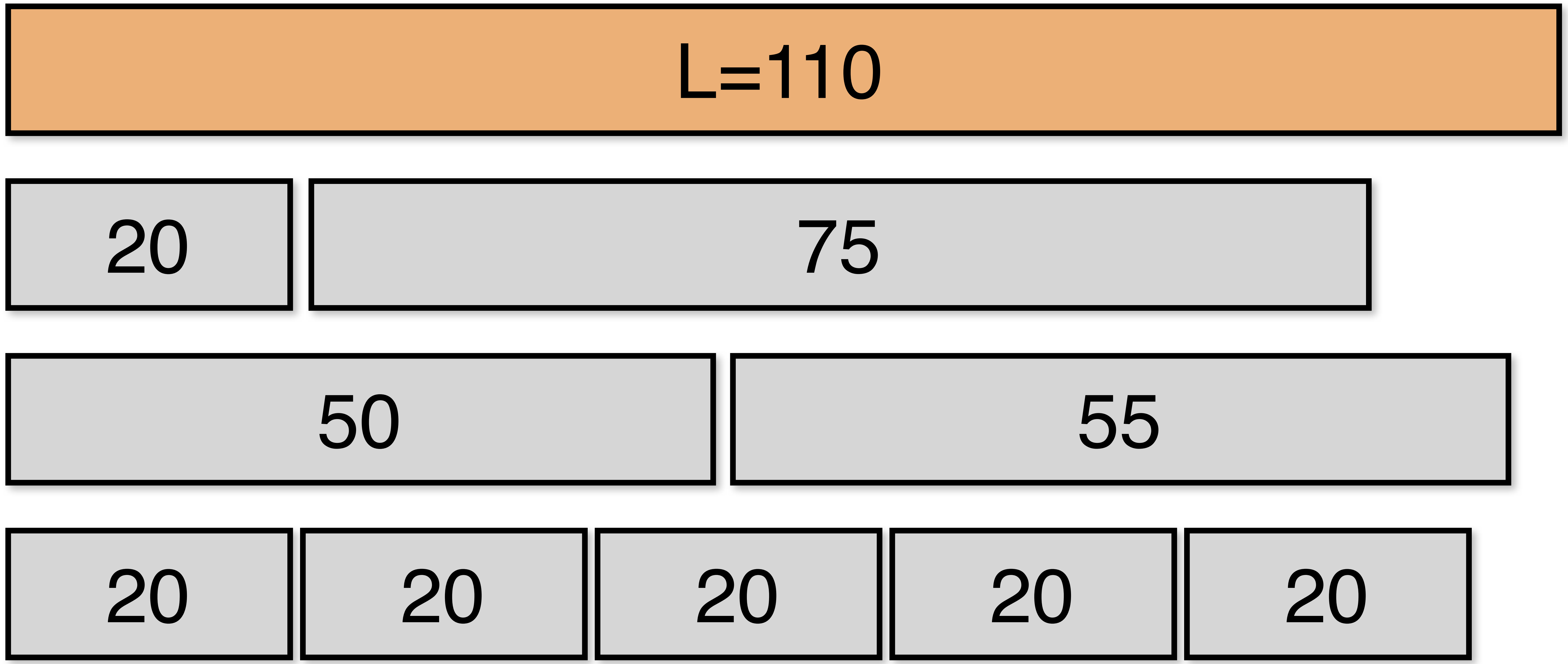

A First MIP Model

- Decision variables
  - $y_b = 1$ if board b is used in the solution
  - x_{sb} is the number of shelves of type s cut from board b

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- Objectives
  - minimize the number of boards used

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\begin{aligned}
\min \quad & \sum_{b \in B} y_b \\
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& M y_b \geq x_{s,b} && b \in B, s \in S \\
& \sum_{s \in S} l_s x_{s,b} \leq L && b \in B \\
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- Is this a good model?
  - linear relaxation, symmetries

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- Key idea
  - reasoning about cutting configurations, i.e., a specific way to cut a board

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- Decision variables
  - x_c: the number of configurations of type c

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- No symmetries
  - reasoning about the numbers of configurations

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- What about enumerate them all?
  - in practical applications, there may be billions and billions
- Can we generate them on demand?

The MIP Program

	X_1	X_1	...	X_i	Demand
Obj	1	1	...	1	
$Self_1$	$n_{1,1}$	$n_{2,1}$	...	$n_{i,1}$	d_1
$Self_2$	$n_{1,2}$	$n_{2,2}$	...	$n_{i,2}$	d_2
...	...	...	...	...	...
$Self_{\|S\|}$	$n_{1,\|S\|}$	$n_{2,\|S\|}$	...	$n_{i,\|S\|}$	$d_{\|S\|}$

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$Self_2$	$n_{1,2}$	$n_{2,2}$	...	$n_{i,2}$	$n_{c,2}$	d_2
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$Self_{\|s\|}$	$n_{1,\|s\|}$	$n_{2,\|s\|}$	...	$n_{i,\|s\|}$	$n_{c,\|s\|}$	$d_{\|s\|}$

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$$cx = c_B A_B^{-1} b + (c - c_B A_B^{-1} A)x$$

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$$1 - \Pi (n_{i,1}, \ldots, n_{i,k})^T$$

dual variables

The MIP Program

	X_1	X_1	...	X_i	X_c	Demand	Dual
Obj	1	1	...	1	1		
$Self_1$	$n_{1,1}$	$n_{2,1}$	...	$n_{i,1}$	$n_{c,1}$	d_1	π_1
$Self_2$	$n_{1,2}$	$n_{2,2}$	...	$n_{i,2}$	$n_{c,2}$	d_2	π_2
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$Self_{\|S\|}$	$n_{1,\|s\|}$	$n_{2,\|s\|}$	...	$n_{i,\|s\|}$	$n_{c,\|s\|}$	$d_{\|s\|}$	$\pi_{\|s\|}$

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- Quality

$$1 - \Pi(n_{i,1}, \ldots, n_{i,|S|})^T < 0$$

Configurations and Linear Programming

- A new configuration must satisfy two conditions
  - feasibility
  - quality: i.e., entering the basis
- Solve the following linear program

$$\min \quad 1 - \sum_{s \in S} \Pi_s^* \; n_s$$

$$s.t$$

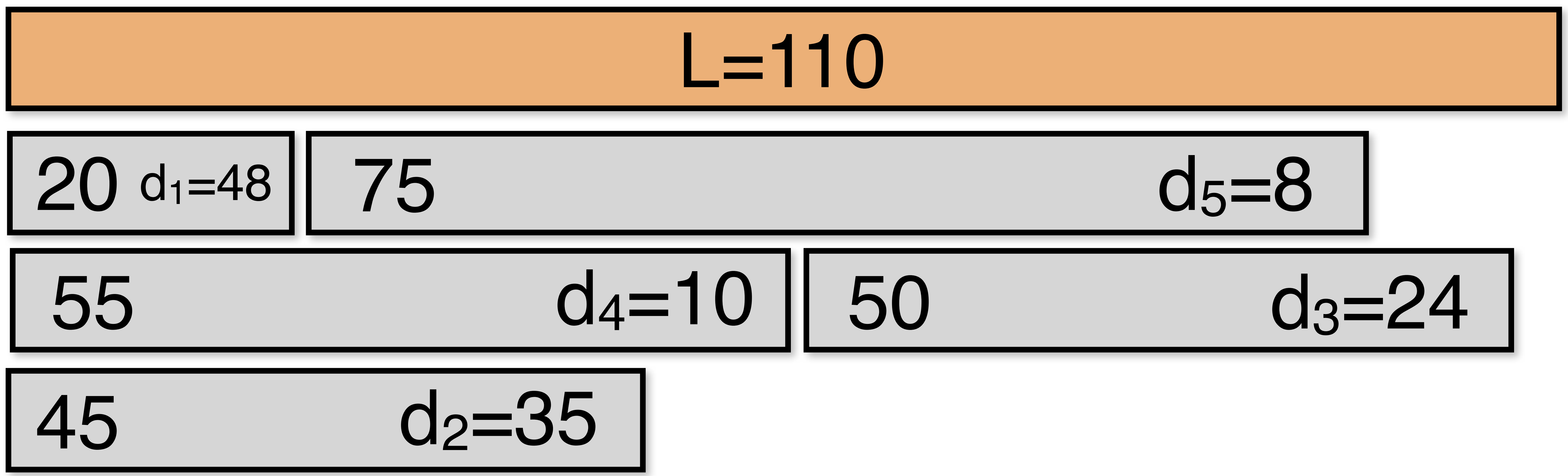
$$\sum_{s \in S} l_s \; n_s \leq L$$

- If the objective is negative, we have a new configuration for the LP relaxation; otherwise, no such configuration exists

Column Generation for Cutting Stock

1. Generate an initial set of configurations
   - one configuration for each shelf type
2. Solve the linear program
   - with the existing configurations
3. Generate a new configuration based on the optimal solution to the relaxation
   - solve the knapsack problem
4. If a new column was found, repeat from step 2
5. Otherwise, round the solution upwards to find an integer solution.

Column Generation in Action



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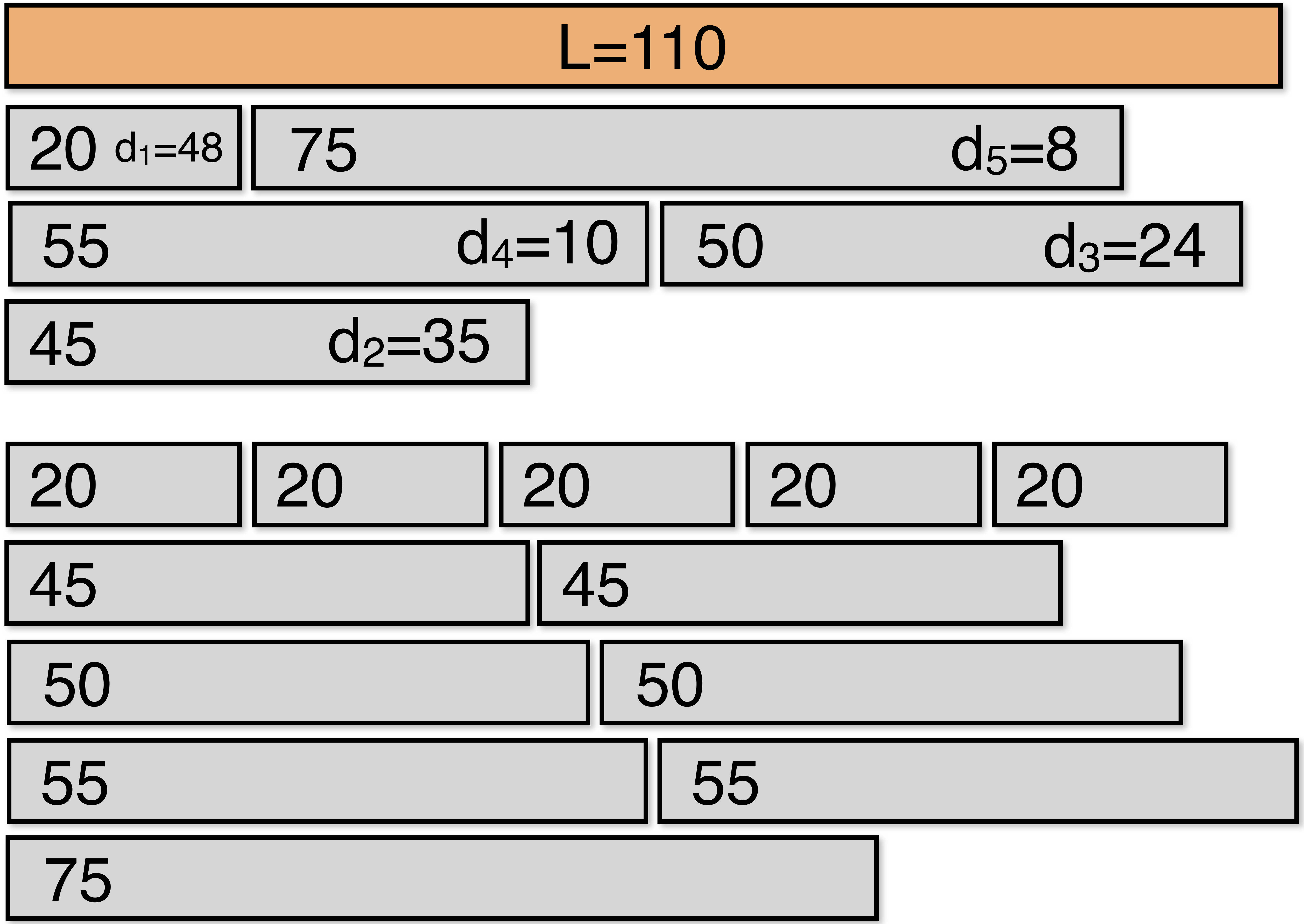

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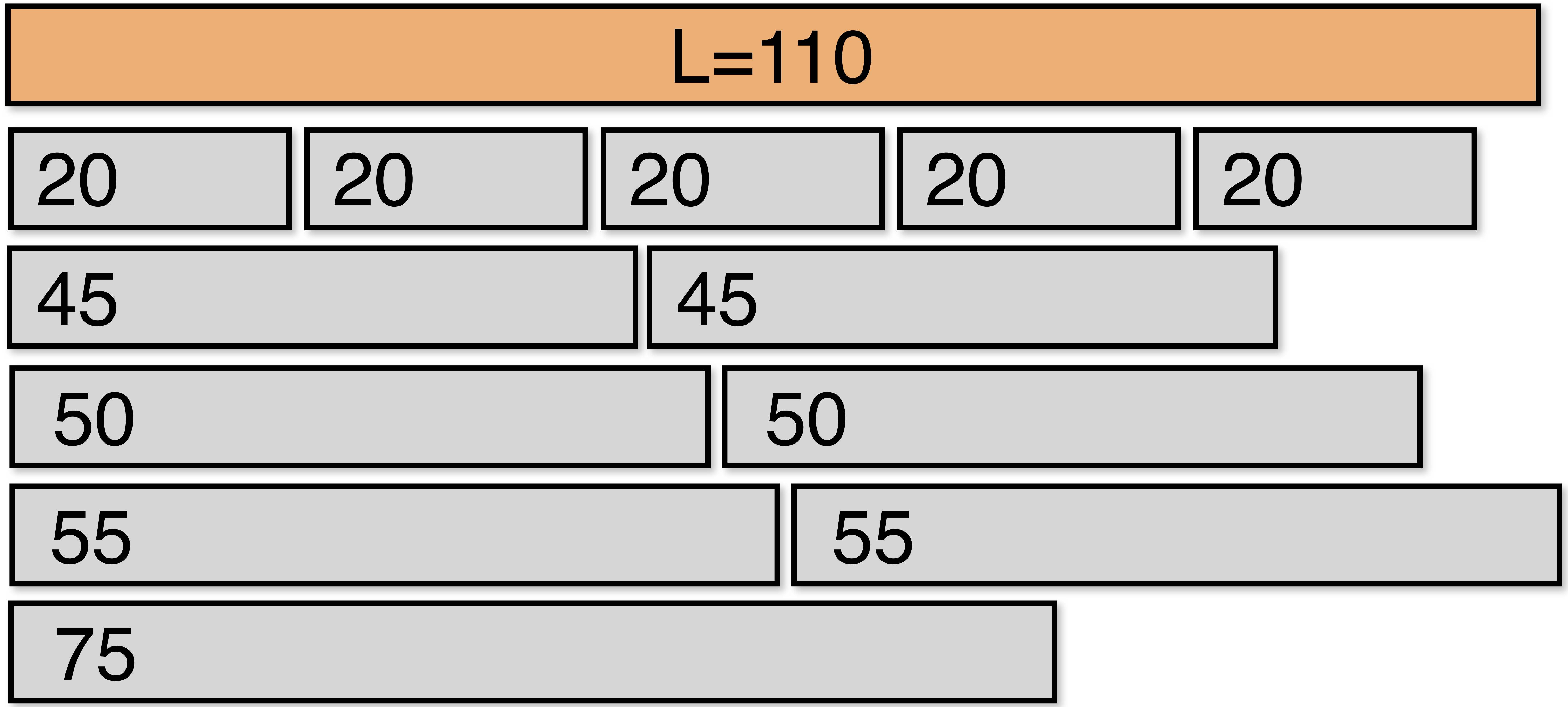

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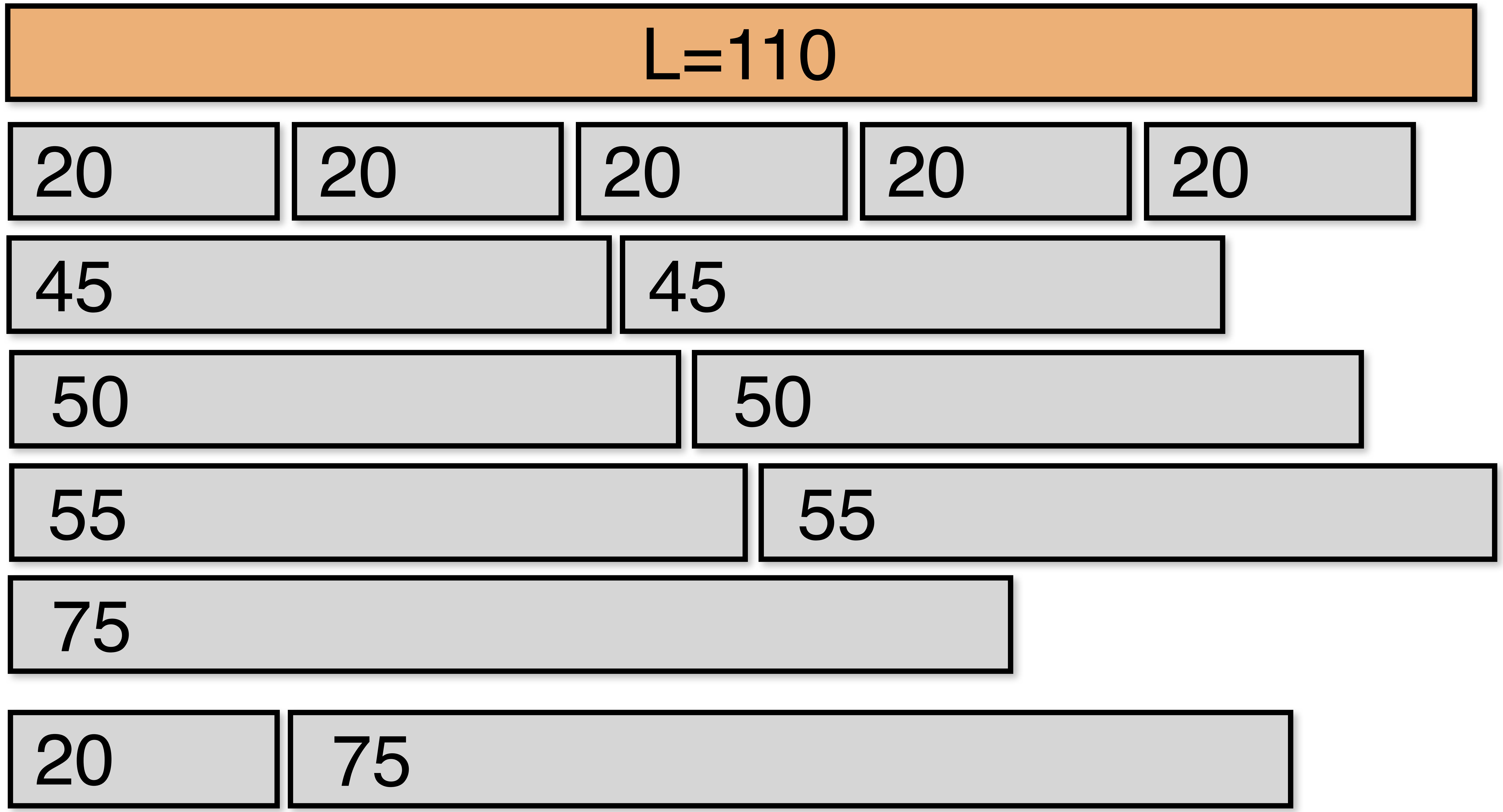

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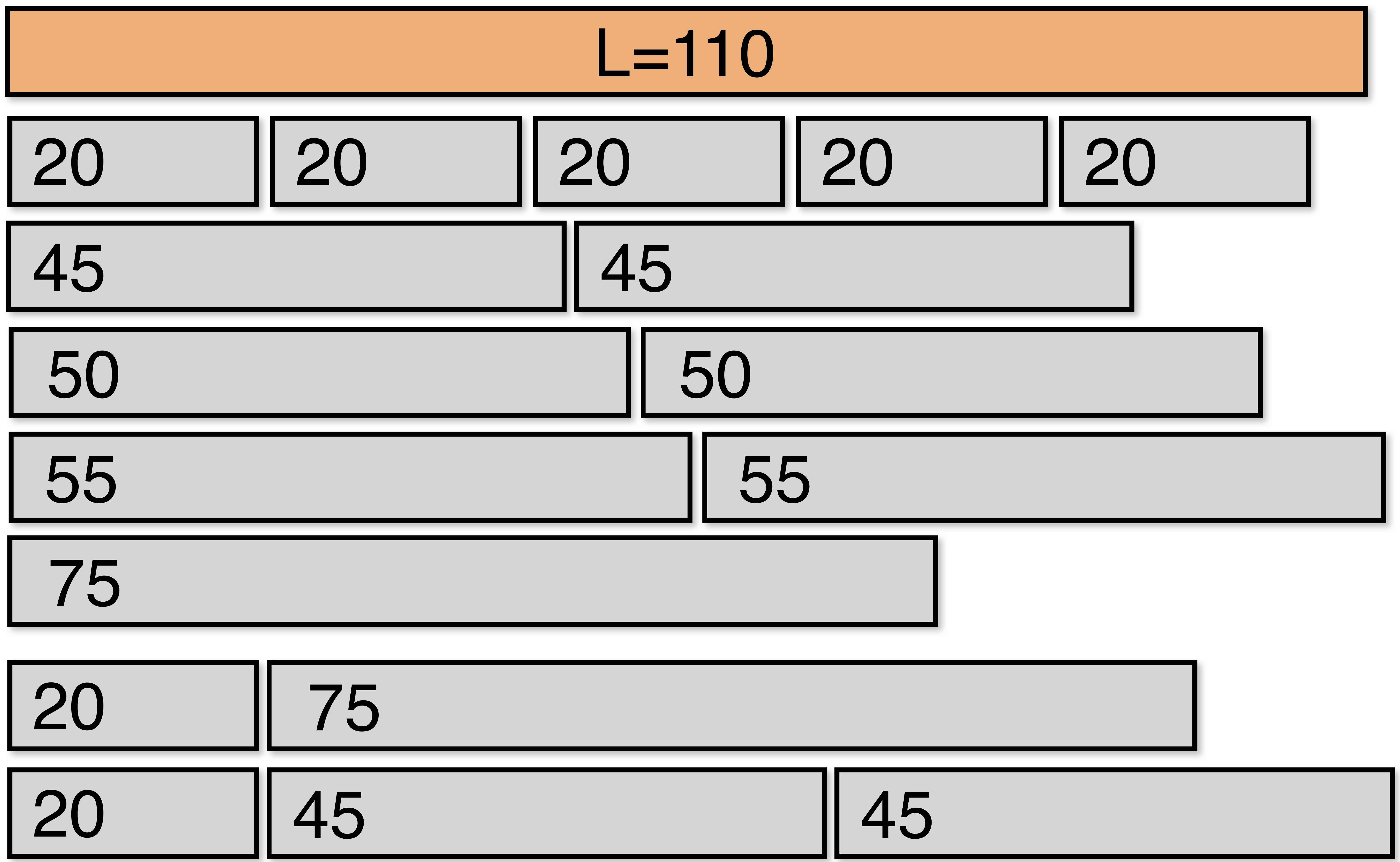

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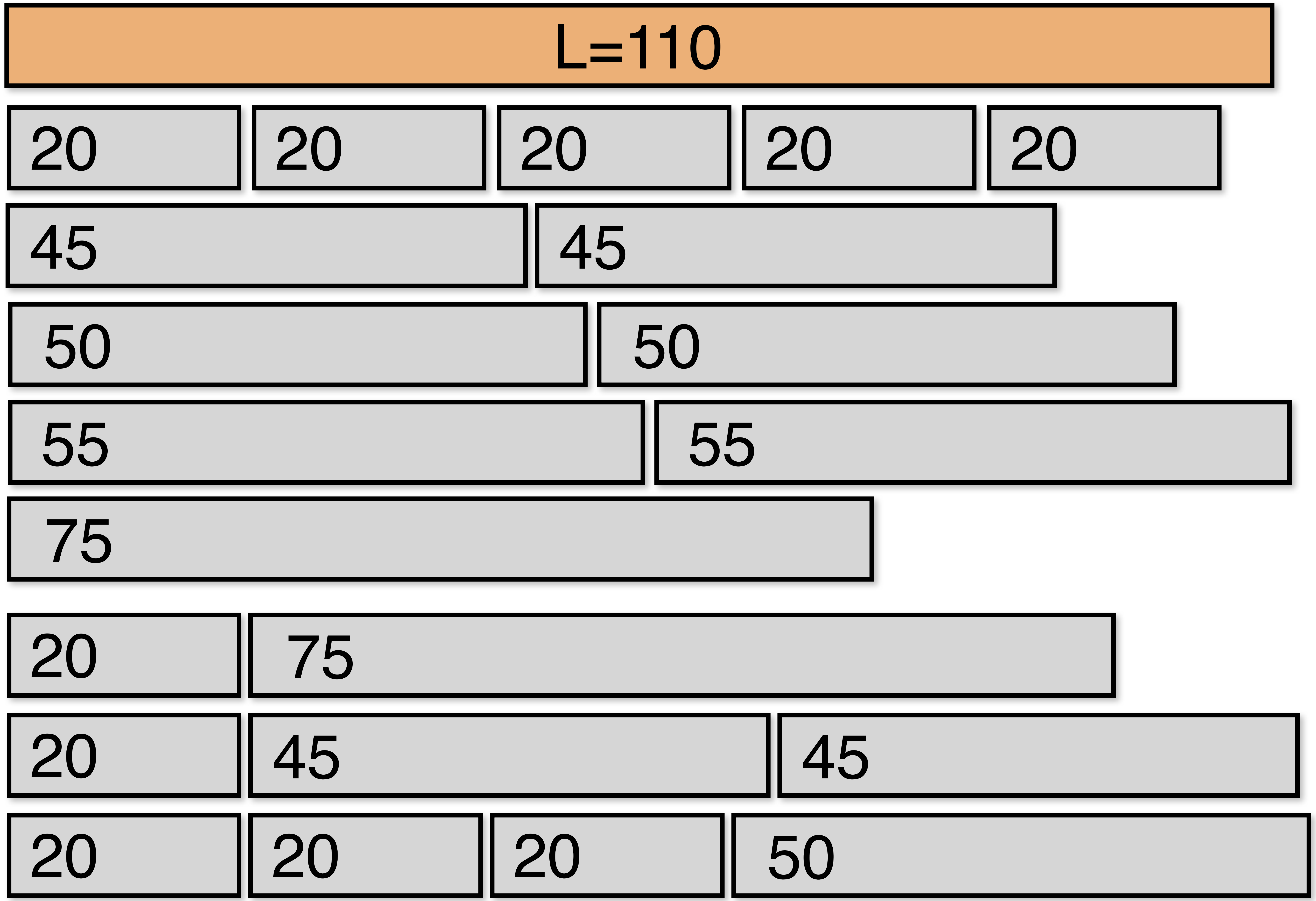

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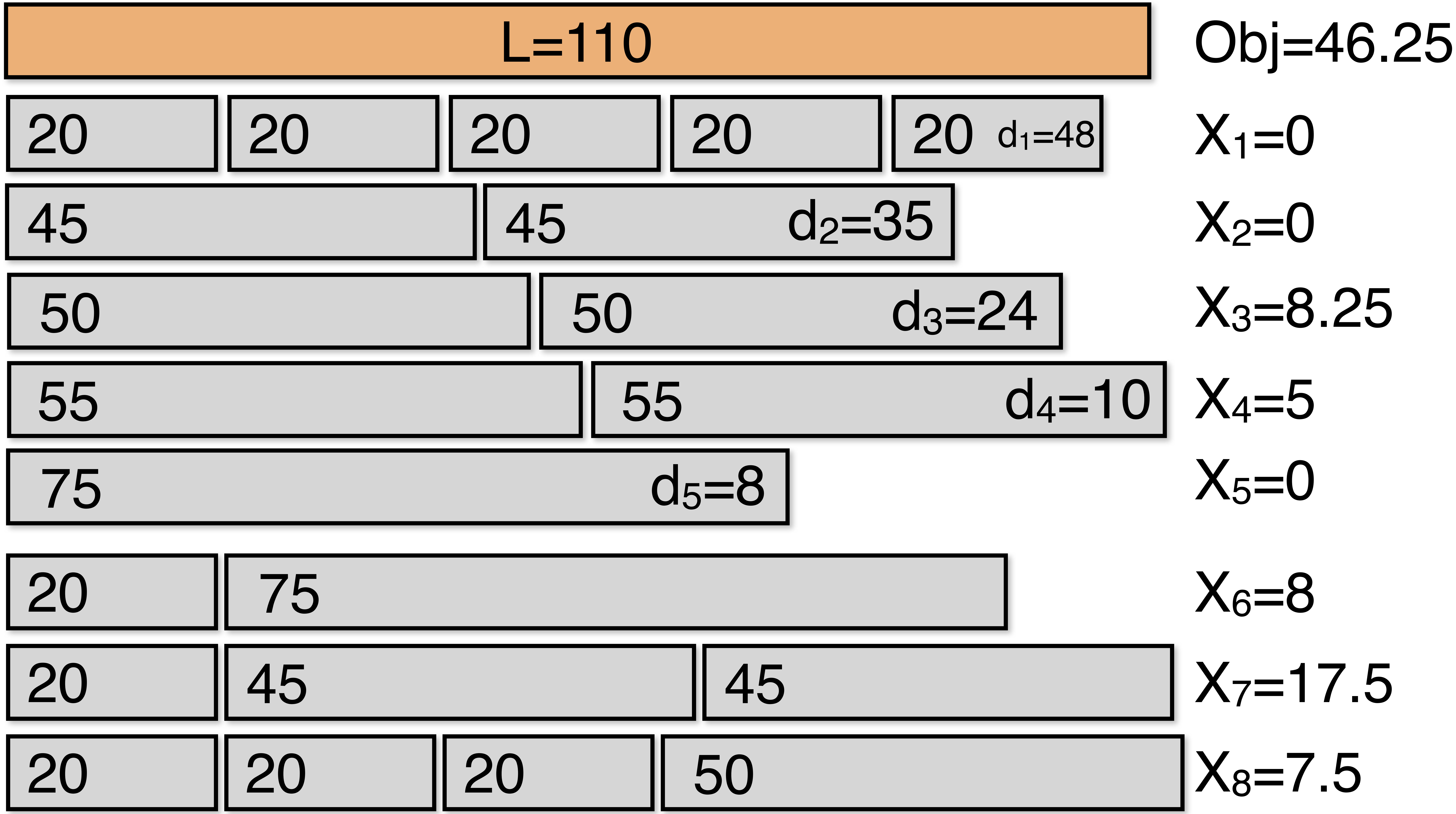

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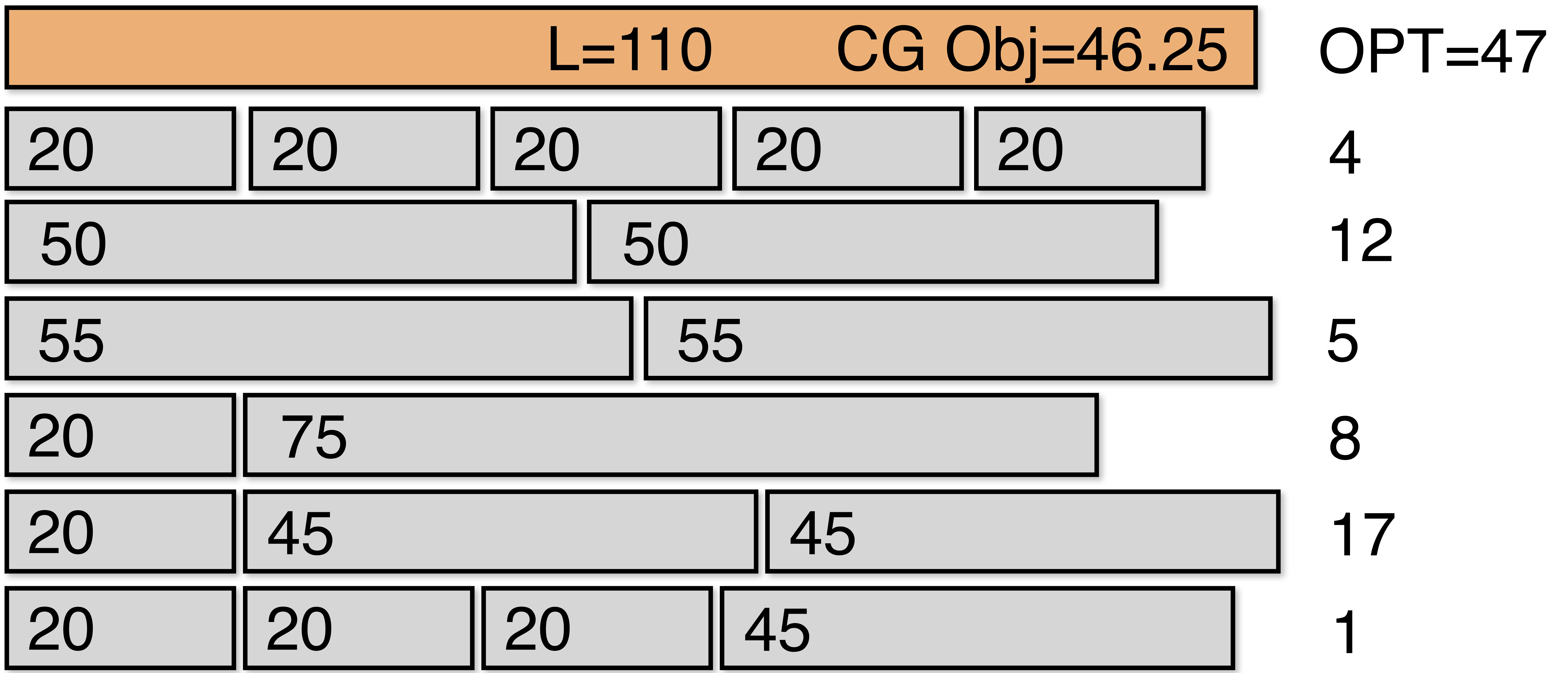

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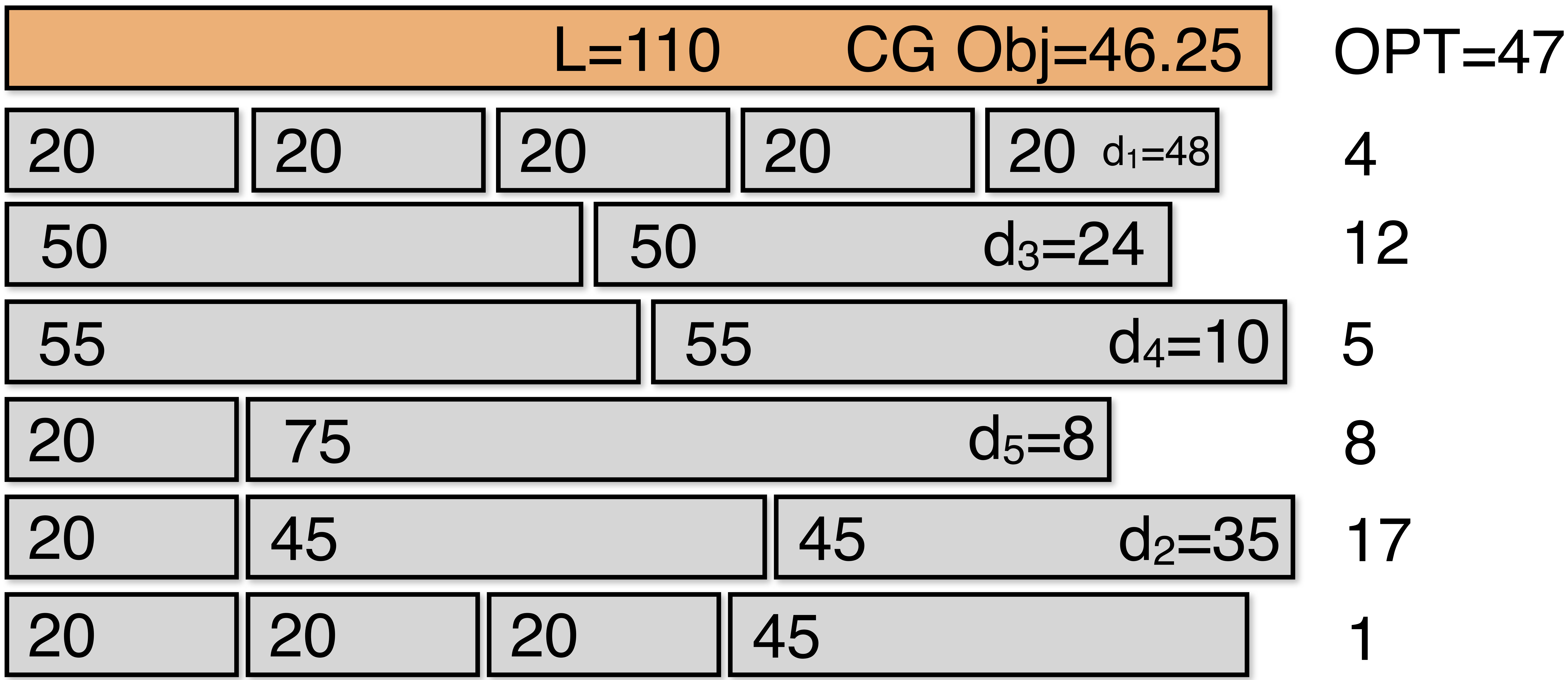

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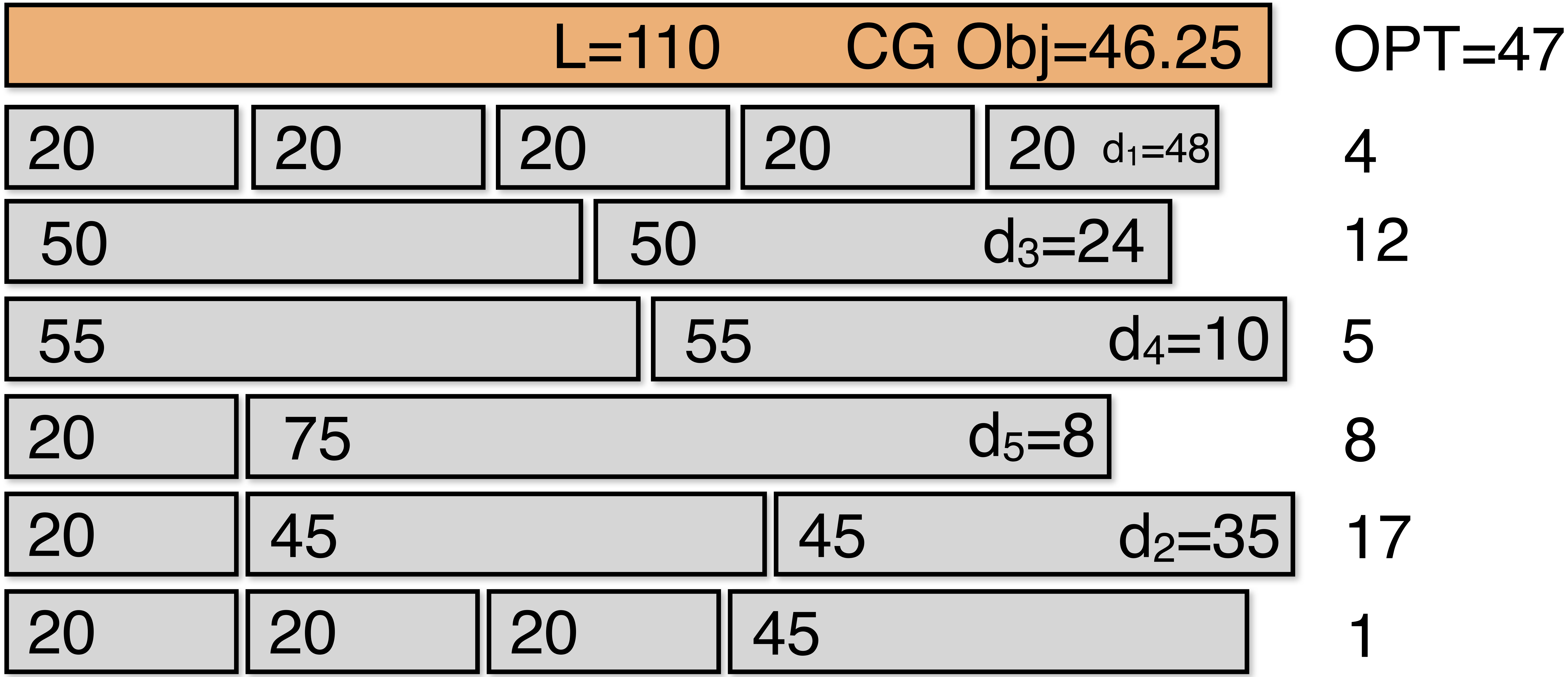

CG Time: 0.03 sec

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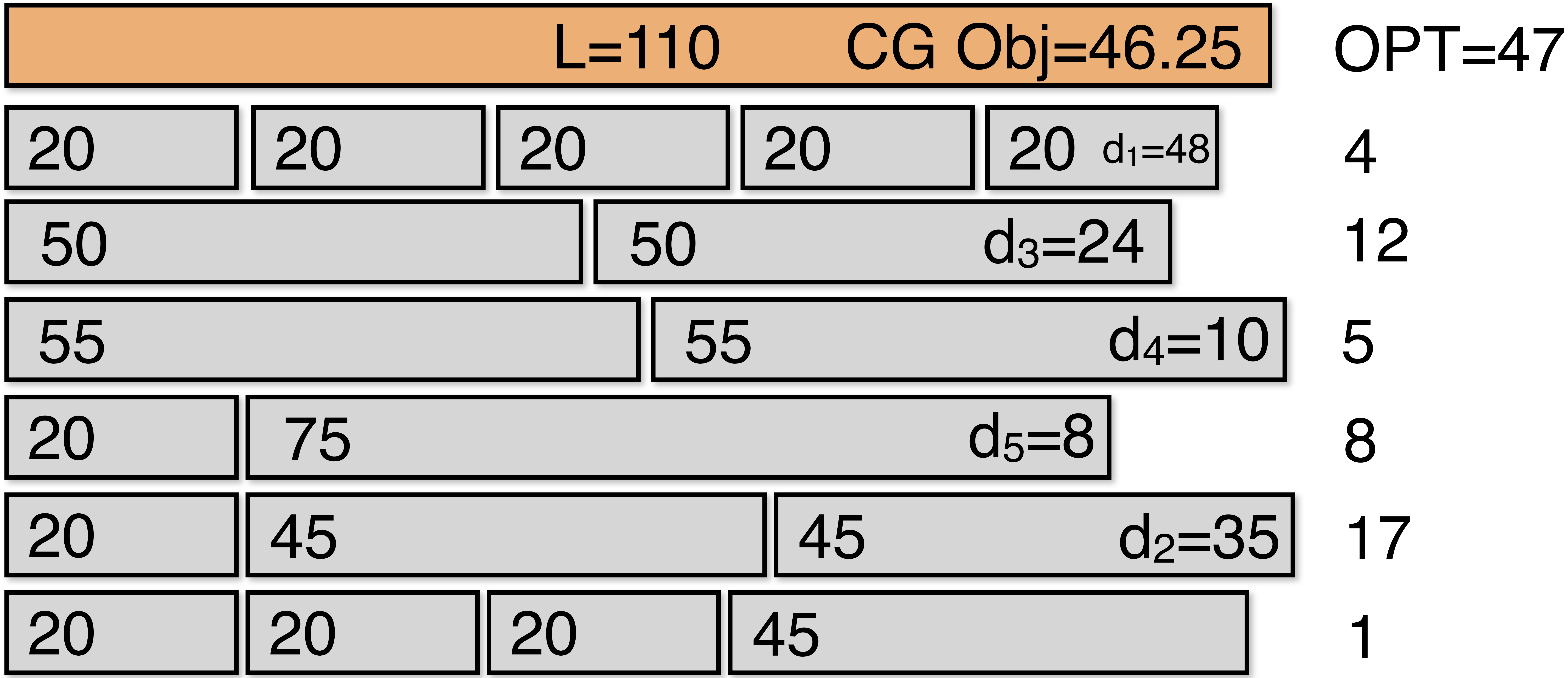

CG Time: 0.03 sec

MIP Time: 4 sec

Branch and Price

- Perform column generation
- If the solution is integral, terminate
- Otherwise, branch and repeat the process on the subproblem
  - generate new columns at the node to obtain a stronger relaxation

Until Next Time