Discrete Optimization

Mixed Integer Programming: Part V

Goals of the Lecture

- Branch and cut
  - Cover cuts
    - separation problem
  - TSP

Cover Cuts

- Consider constraints of the type

$$\sum_{j=1}^{n} a_j x_j \leq b$$

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- Can we find facets for these constraints?
- Cover
  - a set $C \subseteq N = \{1, \ldots, n\}$ is a cover if

$$\sum_{j \in C} a_j > b$$

  - a cover is minimal if $C \setminus \{j\}$ is not a cover for any $j \in C$.

Cover Cuts

- Consider constraints of the type

$$\sum_{j=1}^{n} a_j x_j \leq b$$

- Can we find facets for these constraints?
- If $C \subseteq N = \{1, \ldots, n\}$ is a cover, then

$$\sum_{j \in C} x_j \leq |C| - 1$$

  is a valid inequality.

Cover Cuts

- Consider

$$11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19$$

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- Some minimal cover inequalities

$$x_1 + x_2 + x_3 \leq 2$$

$$x_3 + x_4 + x_5 + x_6 \leq 3$$

Stronger Cover Cuts

- If $C \subseteq N = \{1, \ldots, n\}$ is a cover, then

$$\sum_{j \in E(C)} x_j \leq |C| - 1$$

is a valid inequality

$$\sum_{j=1}^{n} a_j x_j \leq b$$

where

$$E(C) = \{j \mid \forall i \in C : a_j \geq a_i\}$$

Stronger Cover Cuts

- Consider

$$11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19$$

- And

$$x_3 + x_4 + x_5 + x_6 \leq 3$$

Stronger Cover Cuts

- Consider

$$11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19$$

- And

$$x_3 + x_4 + x_5 + x_6 \leq 3$$

- A stronger cover inequality is

$$x_1 + \ldots + x_6 \leq 3$$

Branch and Cut

- Basic idea
  1. formulate the application as a MIP;
  2. solve the linear relaxation; if the linear relaxation is integral, terminate;
  3. find a polyhedral cut which prunes the linear relaxation and is a facet if possible; if you can find such beautiful mathematical object, go back to step 2;
  4. otherwise, settle for the poor man's choice and branch

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The Separation Problem

- Consider a solution x^* to the linear relaxation possibly enhanced by a number of cuts
- We wish to know whether there exists a cover cut that cut x^*

Separation for Cover Cuts

- The cover inequality can be rewritten into

$$\sum_{j \in C} (1 - x_j) \geq 1$$

- Does there exists $C \subseteq N$ that satisfies

$$\sum_{j \in C} (1 - x_j^*) < 1$$

$$\sum_{j \in C} a_j > b$$

Separation for Cover Cuts

- This is equivalent to a beautiful mathematical program

$$\begin{aligned} \min \quad & \sum_{j \in N} (1 - x_j^*) z_j \\ s.t. \\ & \sum_{j \in N} a_j z_j > b \\ & z_j \in \{0, 1\} \end{aligned}$$

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s.t. & \\
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\end{aligned}
$$

- If the minimum value is lower than 1, then we have a cut! All the variables assigned to 1 are a cover.

Separation for Cover Cuts

- Consider the constraint

$$45x_1 + 46x_2 + 79x_3 + 54x_4 + 53x_5 + 125x_6 \leq 178$$

- And the fractional solution

$$x^* = (0, 0, \frac{3}{4}, \frac{1}{2}, 1, 0)$$

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- The separation problem is

$$\begin{array}{lllllllll} min & z_1 & +z_2 & +\frac{1}{4}z_3 & +\frac{1}{2}z_4 & & +z_6 & \\ s.t & & & & & & & \\ & 45z_1 & +46z_2 & +79z_3 & +54z_4 & +53z_5 & +125z_6 & > 178 \end{array}$$

Separation for Cover Cuts

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- Does this remind you of something?

Separation for Cover Cuts

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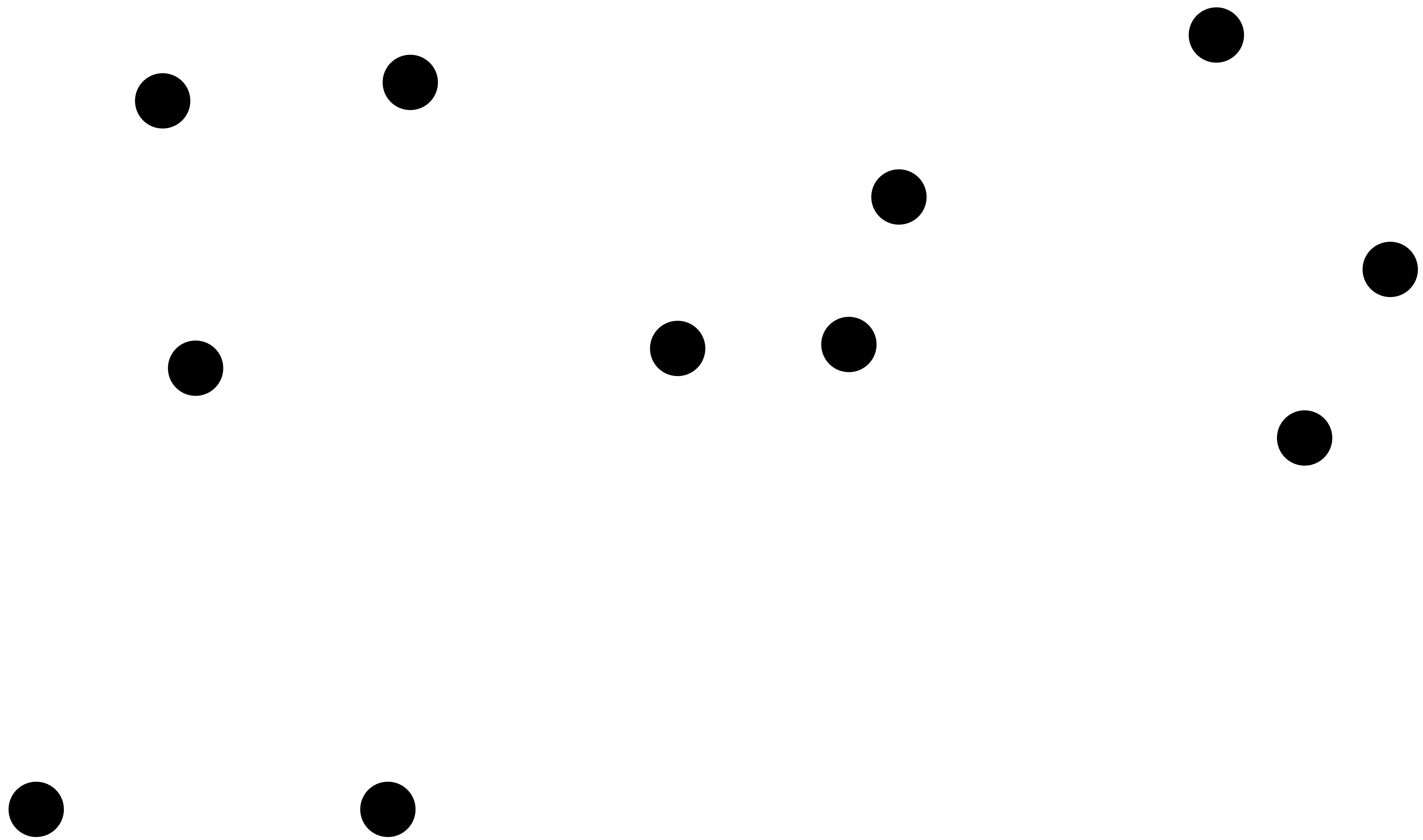
& z_j \in \{0, 1\}
\end{aligned}
$$

- Does this remind you of something?
  – replace z_j by $(1 - y_j)$

Traveling Salesman Problem



MIP for TSP

- How to express the TSP as a MIP?
  - decision variables, constraints, objectives
  - several models obviously

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- How to express the TSP as a MIP?
  - decision variables, constraints, objectives
  - several models obviously
- Decision variables
  - decide whether an edge is part of the tour
- Constraints
  - degree constraints:
    - if a edge is selected, the nodes of the edge must be present in another edge (a.k.a. each node has exactly two edges)

MIP for TSP

- Decision variables
  - x_e is 1 if edge e is in the solution
- Notations
  - V is the set of vertices
  - E is the set of edges
  - δ(v): edges adjacent to vertex v
  - δ(S): edges with exactly one vertex in S ⊆V
  - γ(S): edges with both vertices in S ⊆V
  - $x_{\{e_1,\ldots,e_n\}} = x_{e_1} + \ldots + x_{e_n}$

MIP for TSP

$$\begin{array}{lll} \min & \displaystyle\sum_{e \in E} c_e x_e & \\ \text{subject to} & & \\ & x_{\delta(v)} = 2 & v \in V \\ & x_e \in \{0,1\} & e \in E \end{array}$$

MIP for TSP

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minimize cost

MIP for TSP

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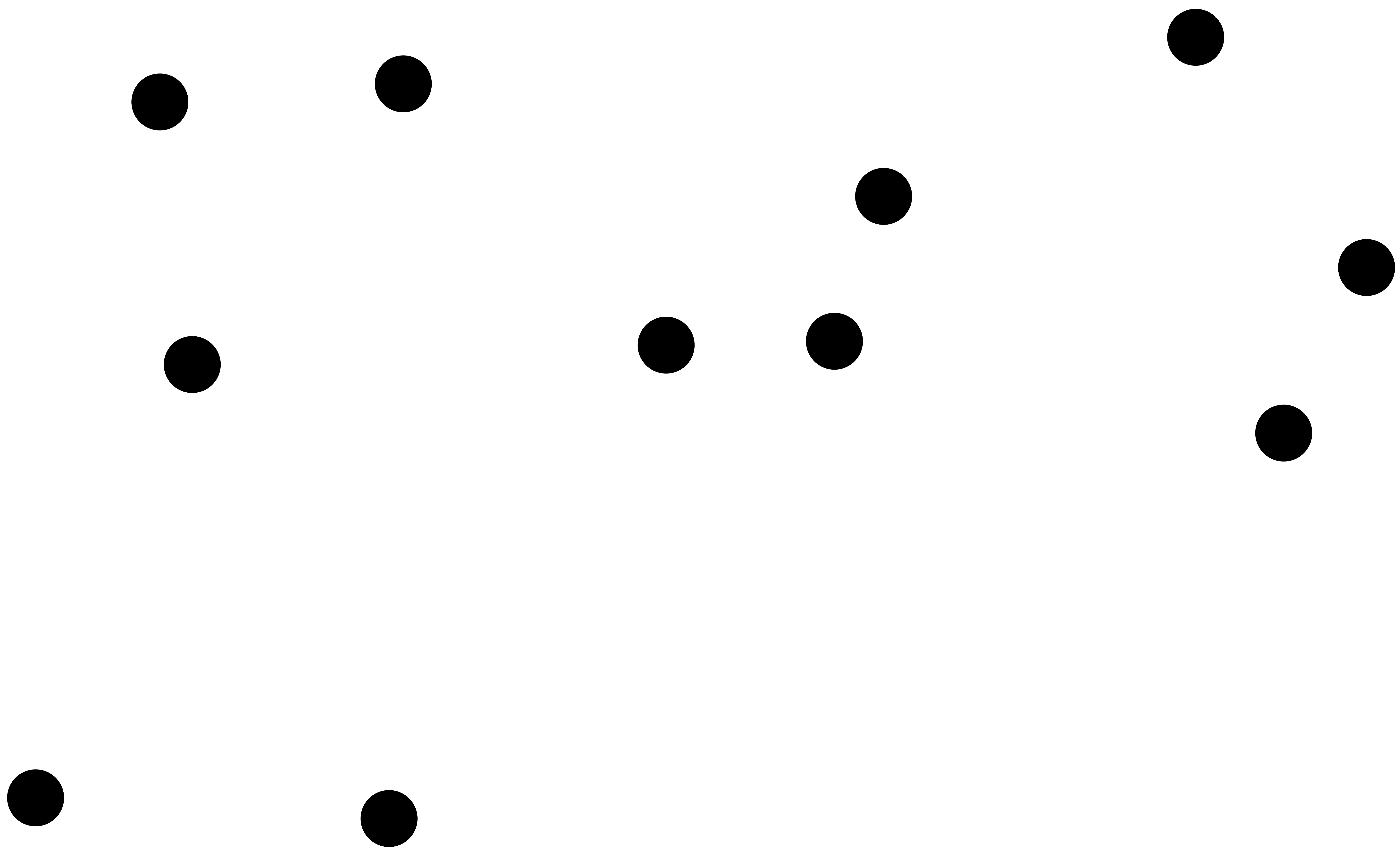
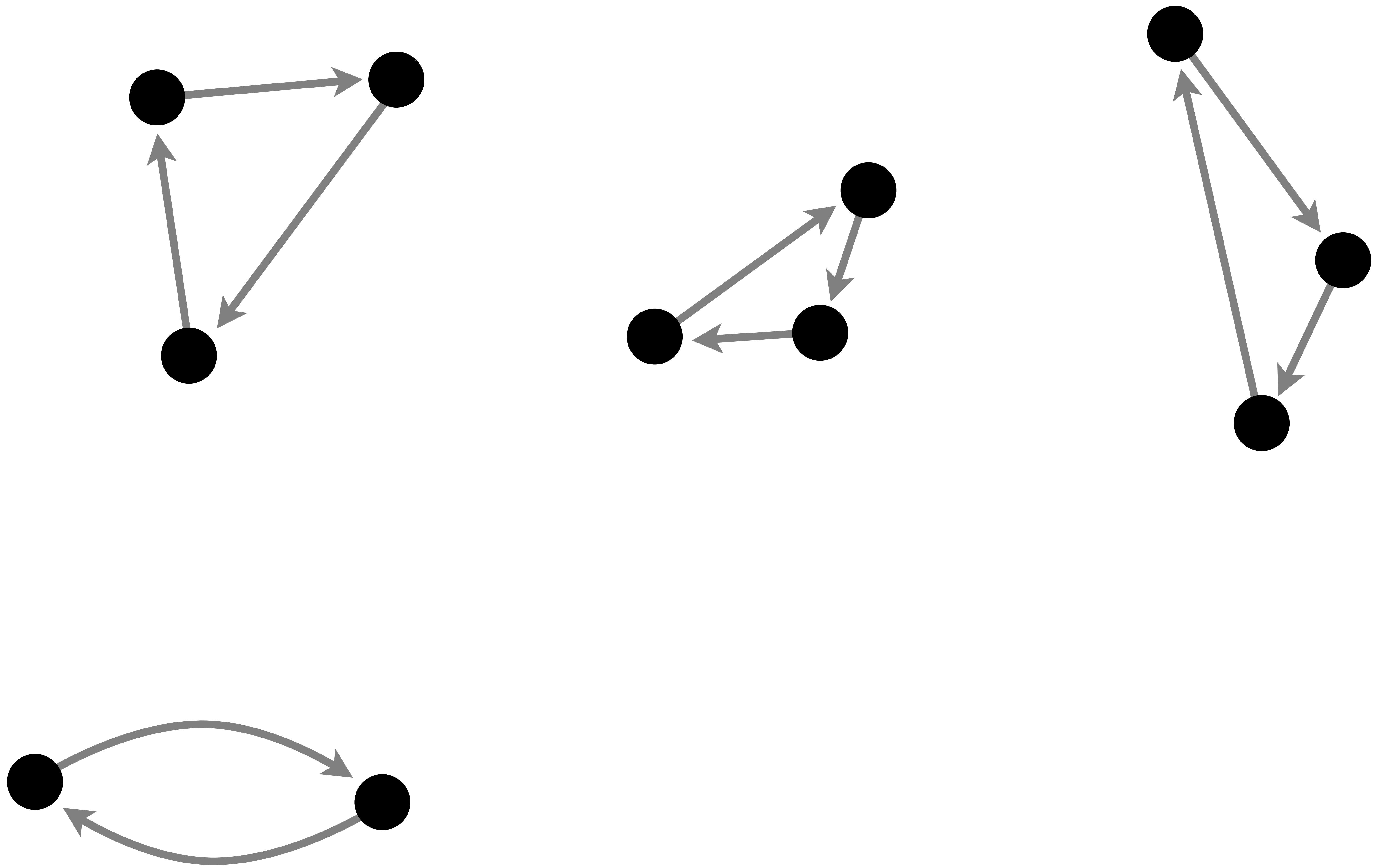
flow conservation

minimize cost

Traveling Salesman Problem



The Degree Constraint Formulation



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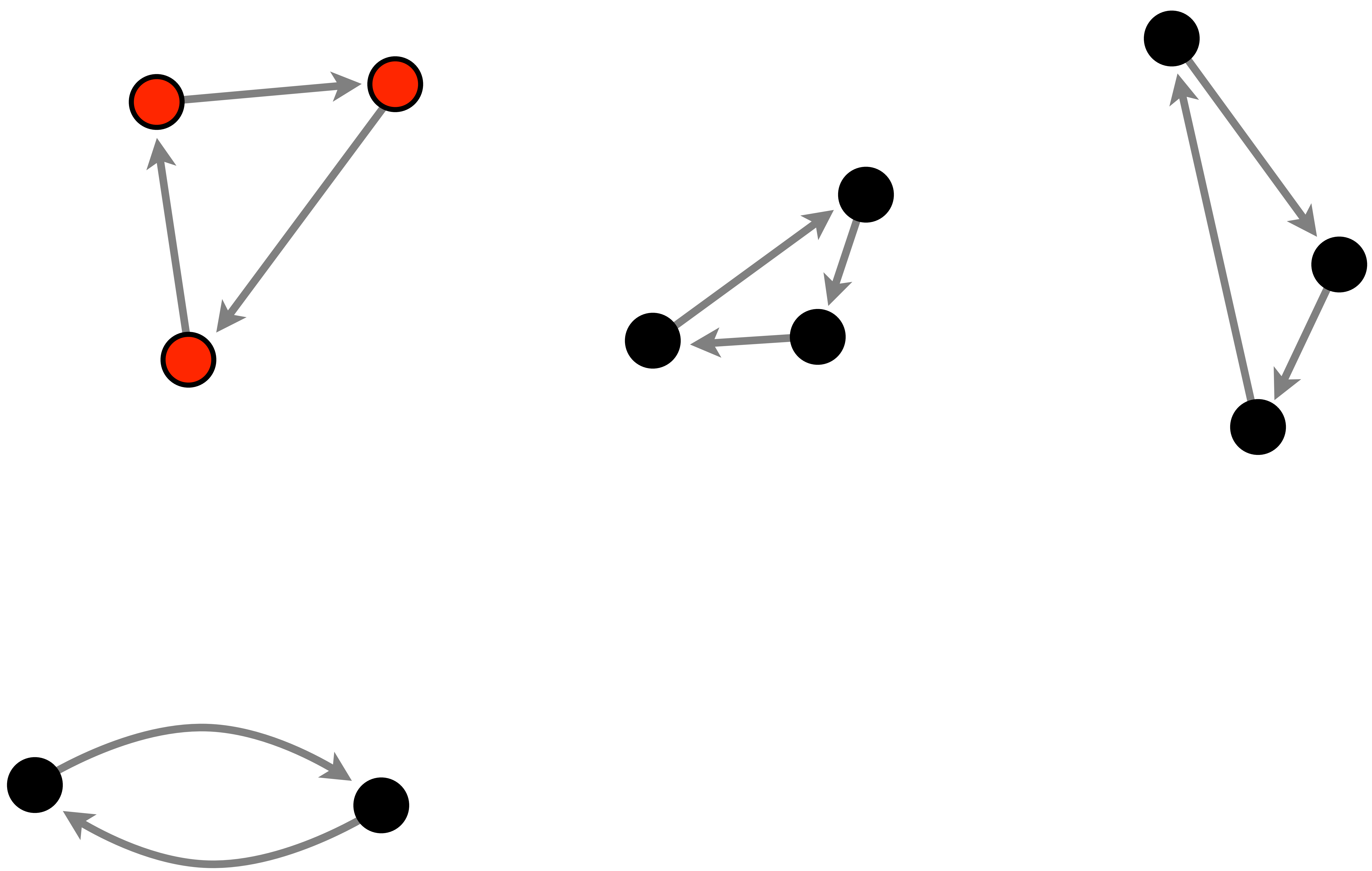

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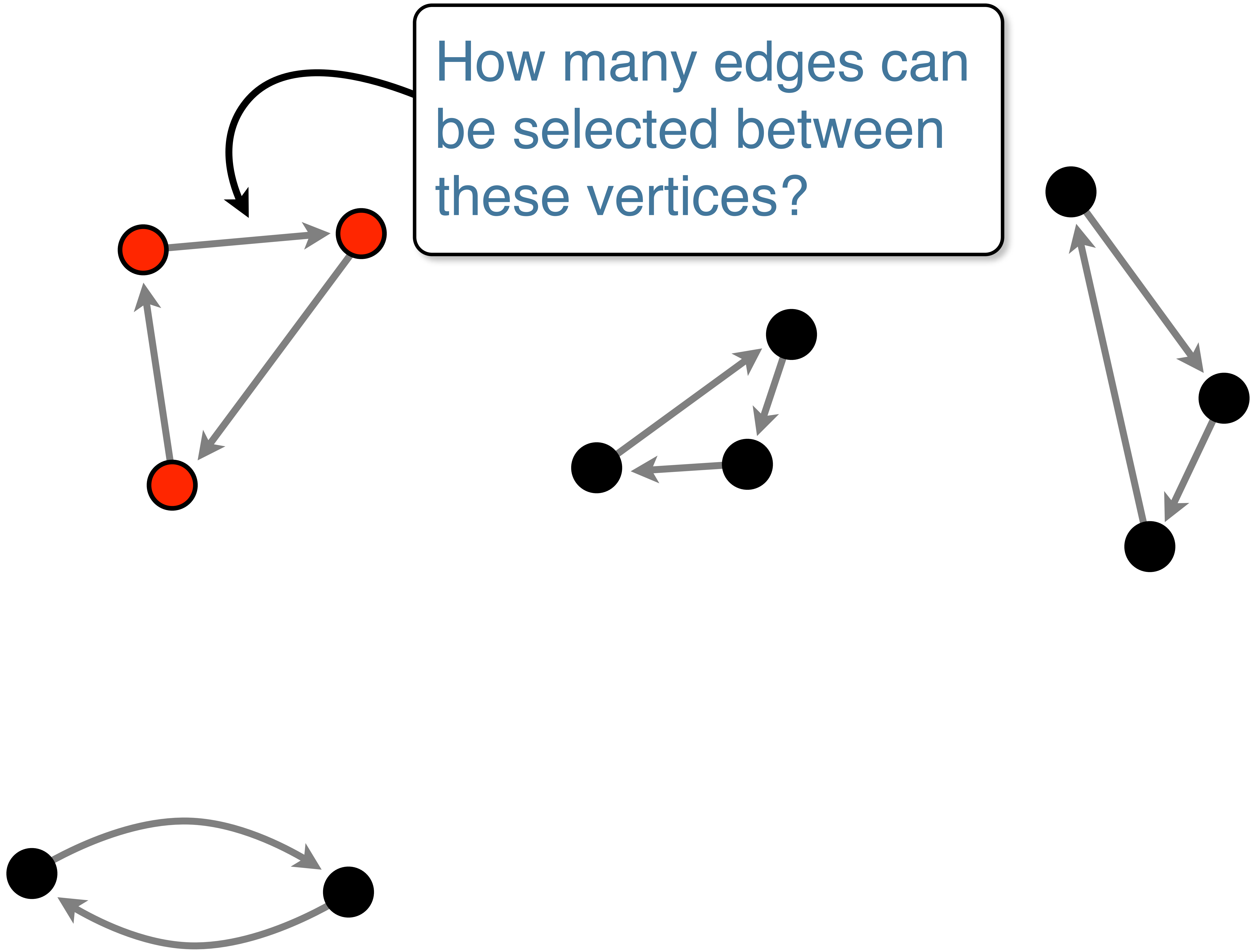

The Subtour Elimination Problem

$$
\begin{array}{lll}
\min & \displaystyle\sum_{e \in E} c_e x_e & \\
\text{subject to} & & \\
& x_{\delta(v)} = 2 & v \in V \\
& x_{\gamma(S)} \leq |S| - 1 & S \subset V \\
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The Subtour Elimination Problem

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- What is the issue with the subtour constraints?
  - there are exponentially many of them

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\begin{array}{lll}
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\text{subject to} & & \\
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$$

- What is the issue with the subtour constraints?
  - there are exponentially many of them
- Branch and cut
  - generate them on demand: separation

The Subtour Elimination Problem

$$\min \quad \sum_{e \in E} c_e x_e$$

subject to

$$\begin{aligned} x_{\delta(v)} &= 2 & v &\in V \\ \boxed{x_{\delta(S)} \geq 2} & & \boxed{S \subset V} \\ x_e &\in \{0,1\} & e &\in E \end{aligned}$$

- How to separate subtour constraints?

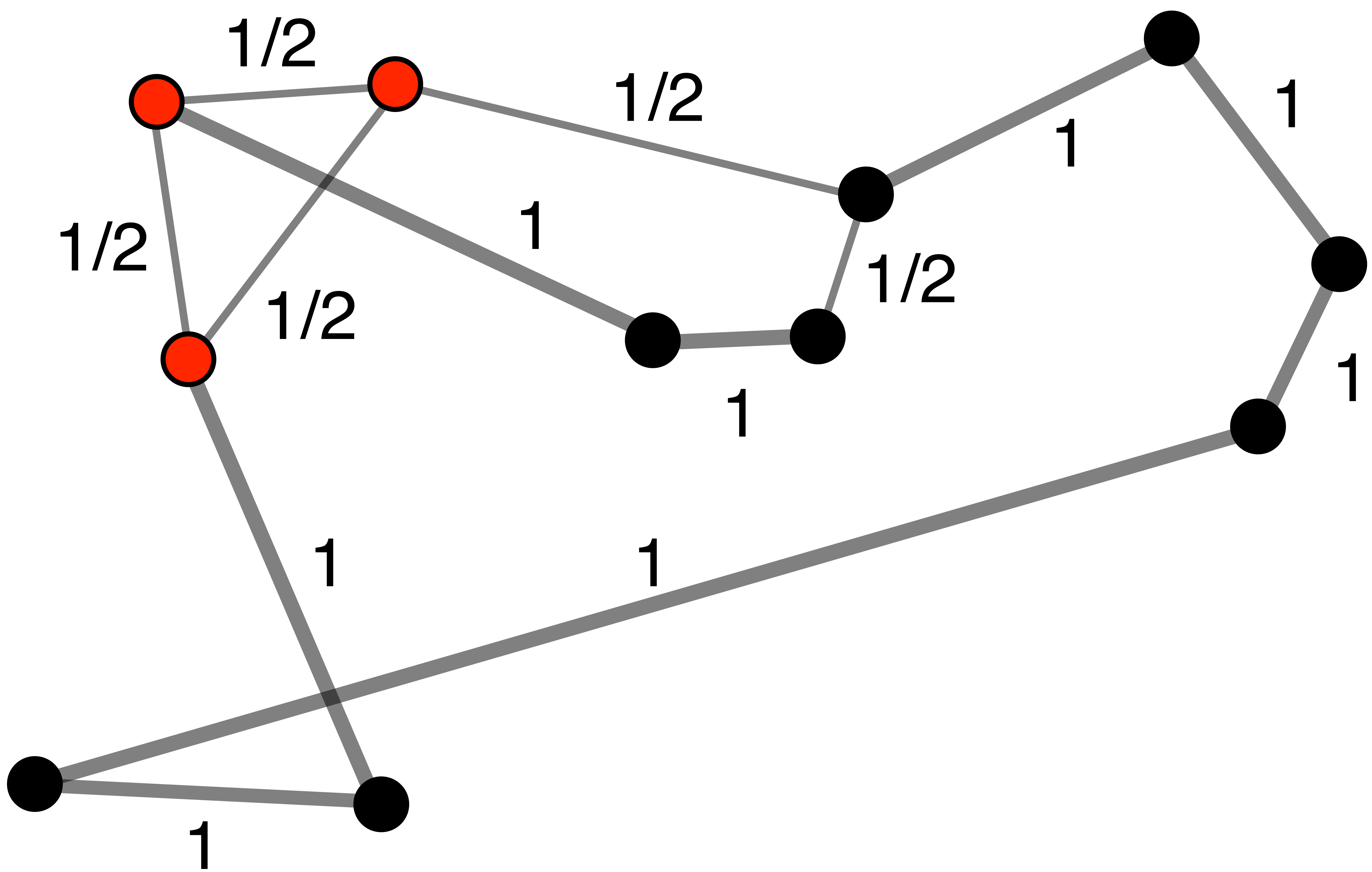
Separation of Subtour Constraints

- Build a graph G* = (V,E) where
  - the weight of edge e is $w(e) = x_e^*$

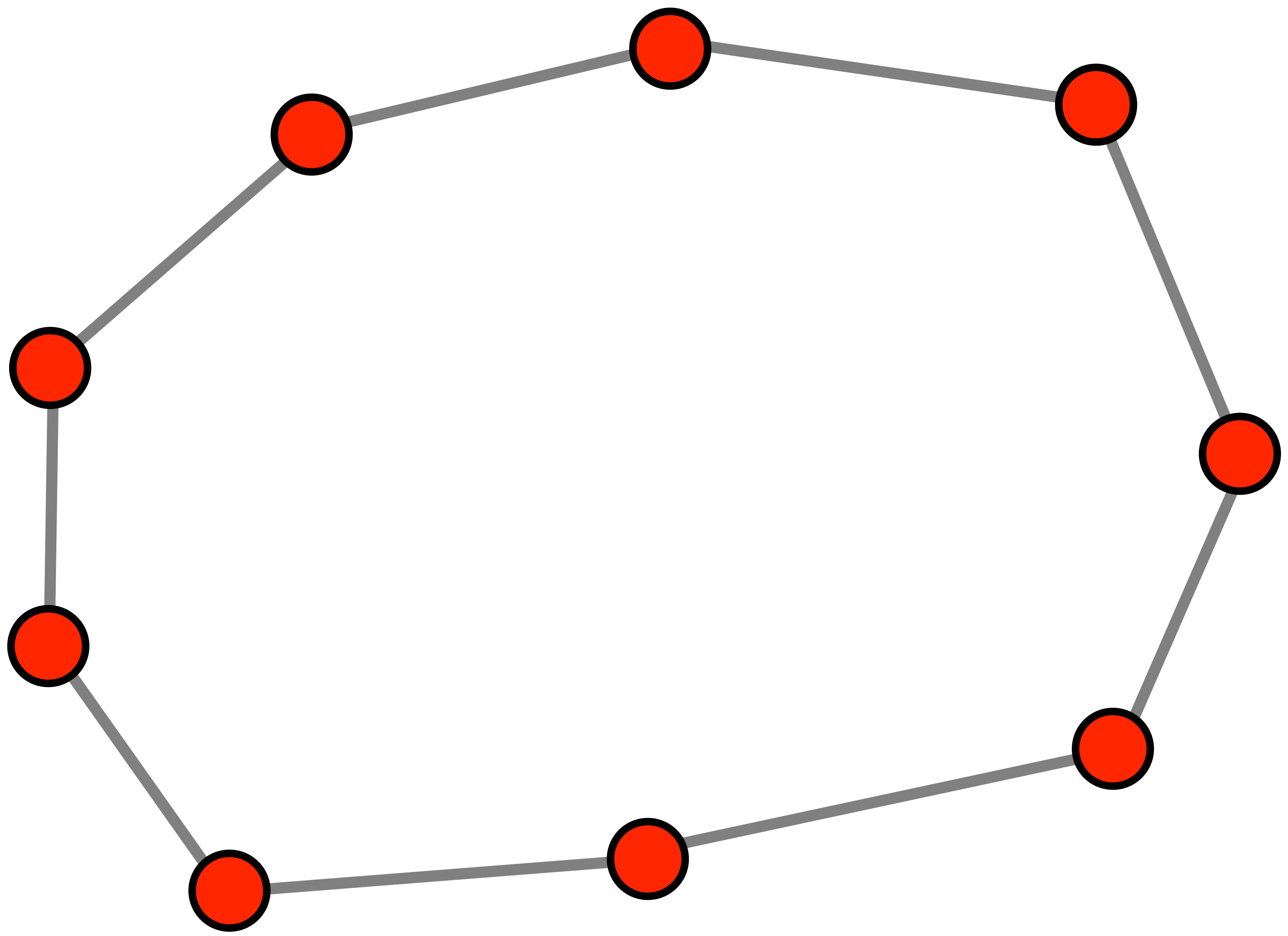
Separation of Subtour Constraints

- Build a graph G* = (V,E) where
  - the weight of edge e is $w(e) = x_e^*$
- Finding a separation consists of finding
  - a minimum cut in G*
  - if the weight of the cut is smaller than 2, then we have isolated a subtour constraint violated by the linear relaxation
  - finding such a cut takes polynomial time

The Subtour Elimination Problem



Comb Constraints



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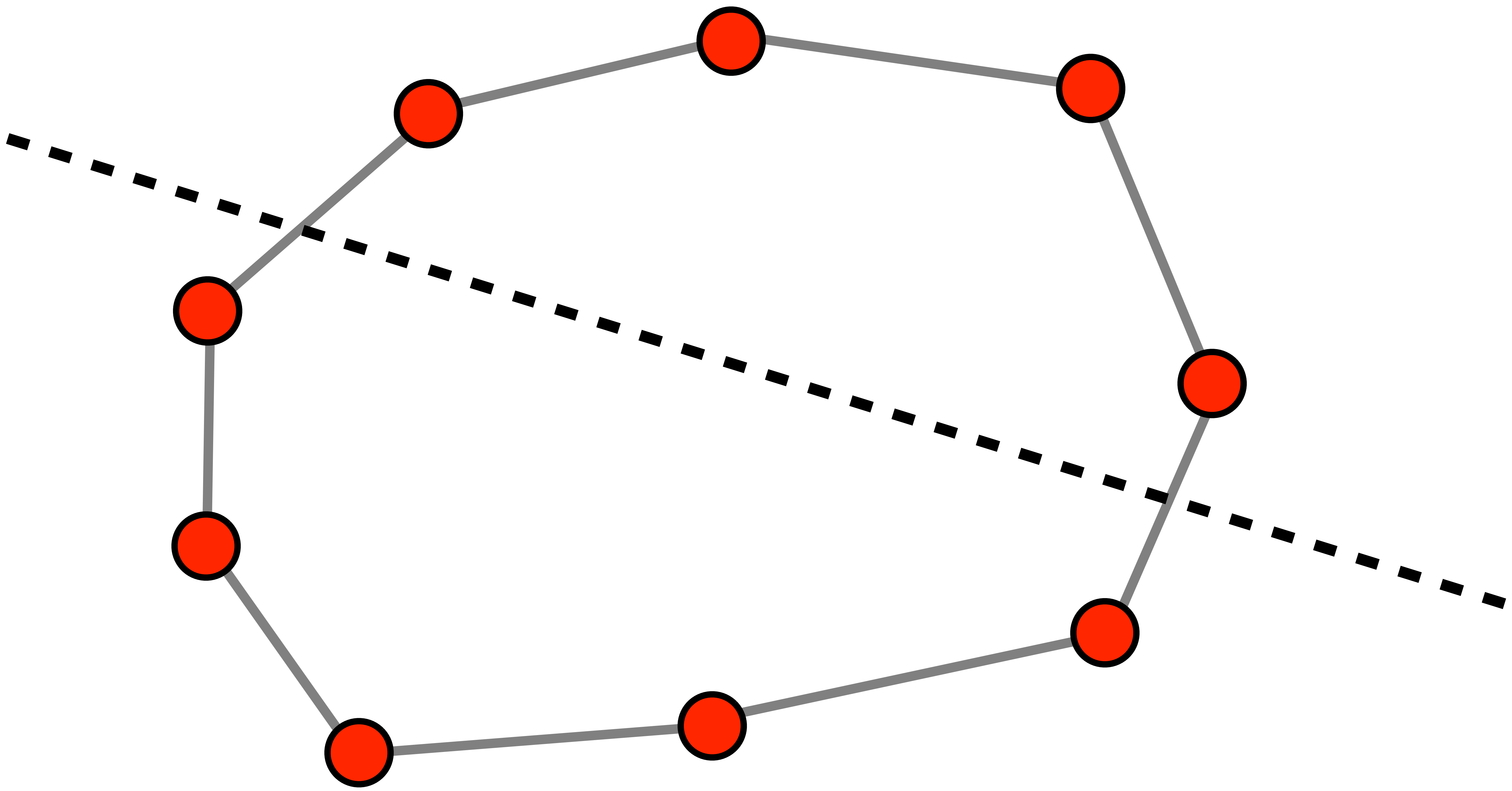
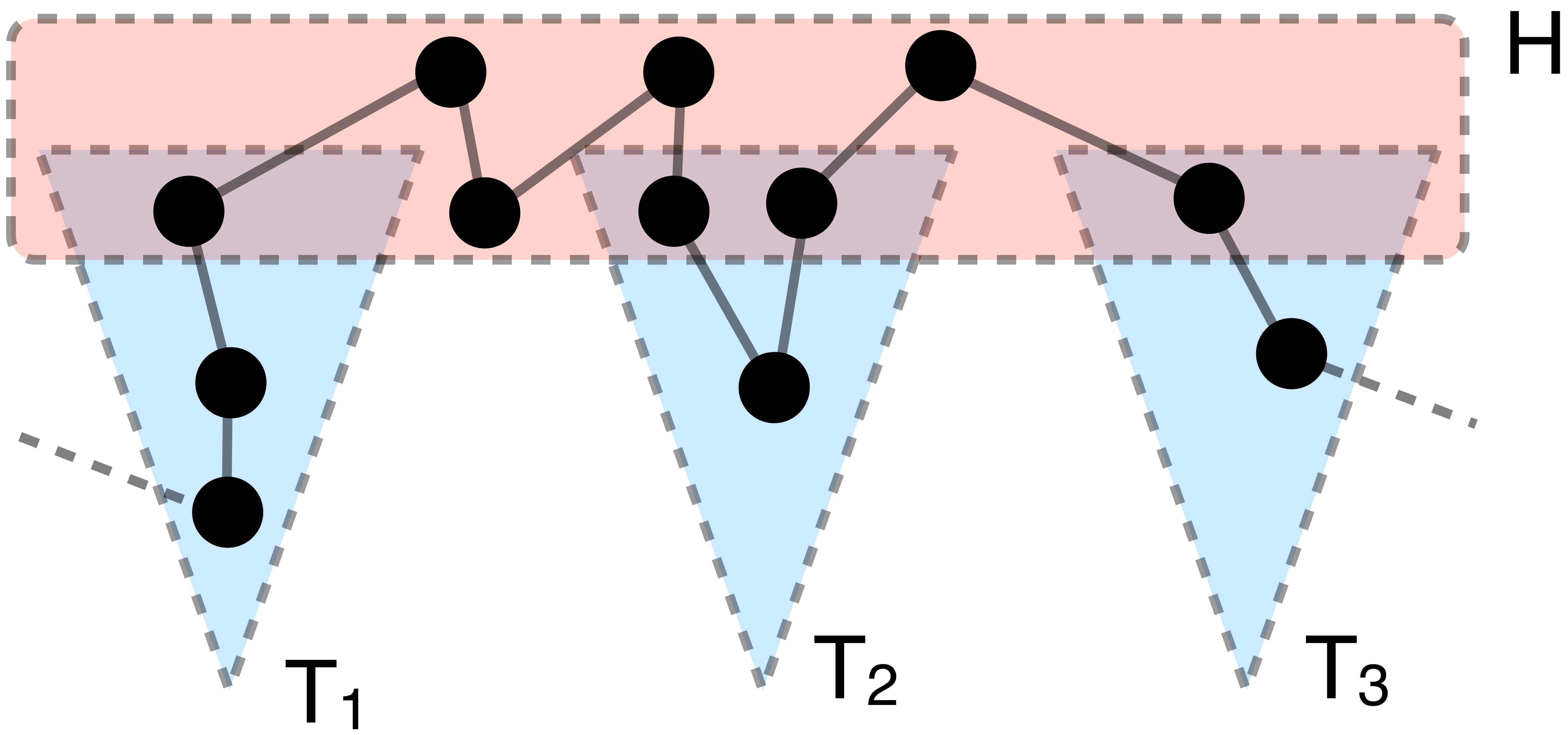


- How many edges do you cross?

Comb Constraints



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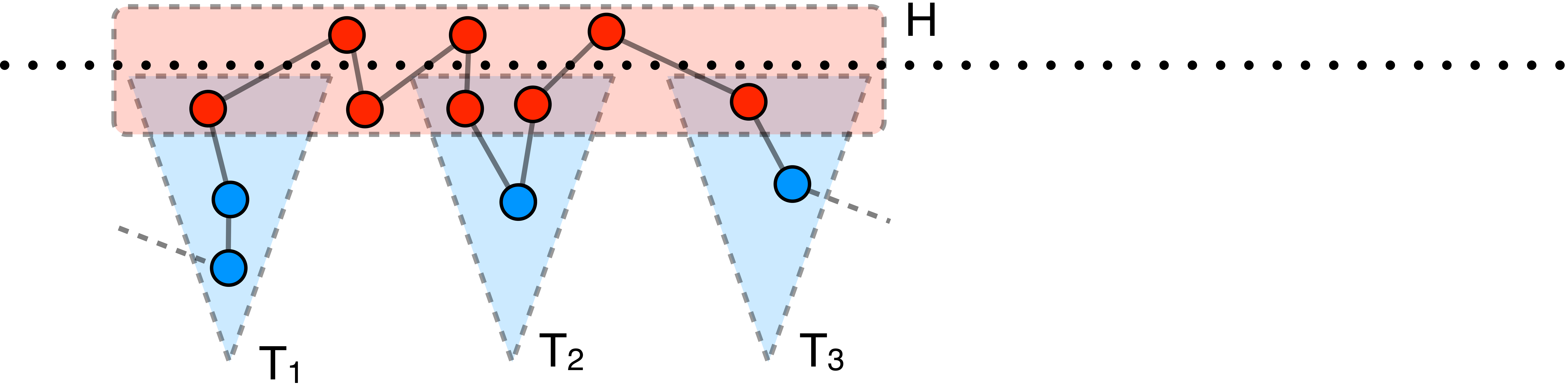

- comb inequalities

$$x_{\gamma(H)} + \sum_{i=1}^{t} x_{\gamma(T_i)} \leq |H| + \sum_{i=1}^{k} |T_i| - \lceil \frac{3k}{2} \rceil$$

Branch and Cut on the TSP

- On benchmarks from the TSPLIB
  - subtour elimination: 2% of optimality gap
  - subtour + comb cuts: 0.5% of optimality gap

- Other cuts are needed on very large instances

Until Next Time