Discrete Optimization

Mixed Integer Programming: Part III

Goals of the Lecture

- Mixed Integer Programming
  - Cutting planes
  - Gomory cuts

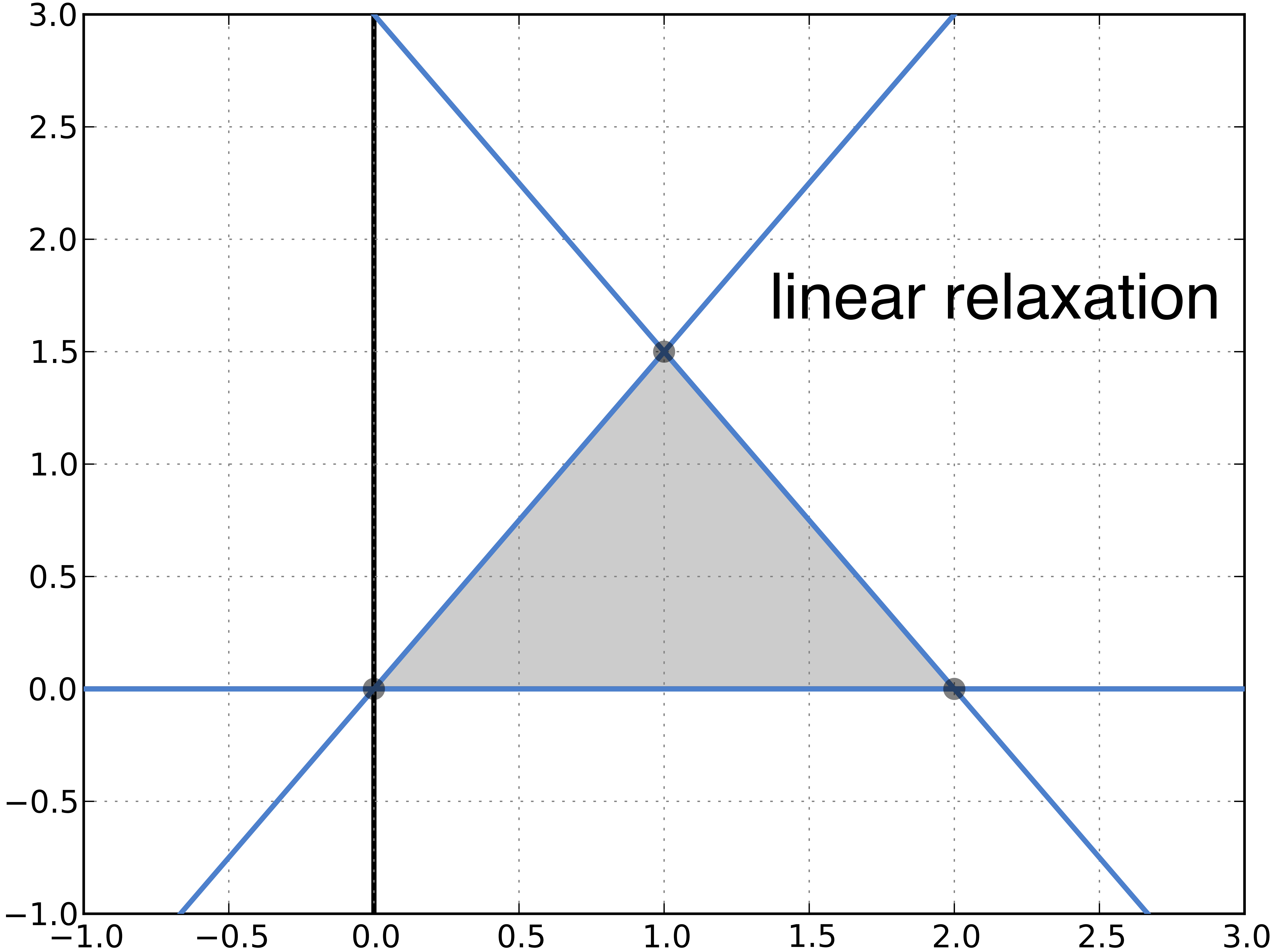
Cutting Planes: Key Idea

- Adding a linear constraint that
  - is valid: i.e., it does not remove any feasible solution
  - helps: i.e., it cuts the optimal solution to the linear relaxation

Cutting Planes: Key Idea

$$
\begin{array}{llllll}
\max & x_2 & & & & \\
\text{subject to} & & & & & \\
 & 3x_1 & + & 2x_2 & \leq & 6 \\
 & -3x_1 & + & 2x_2 & \leq & 0 \\
 & & & x_i & \geq & 0 \\
 & & & x_i & & integer
\end{array}
$$

Cutting Planes at Work (geometrically)



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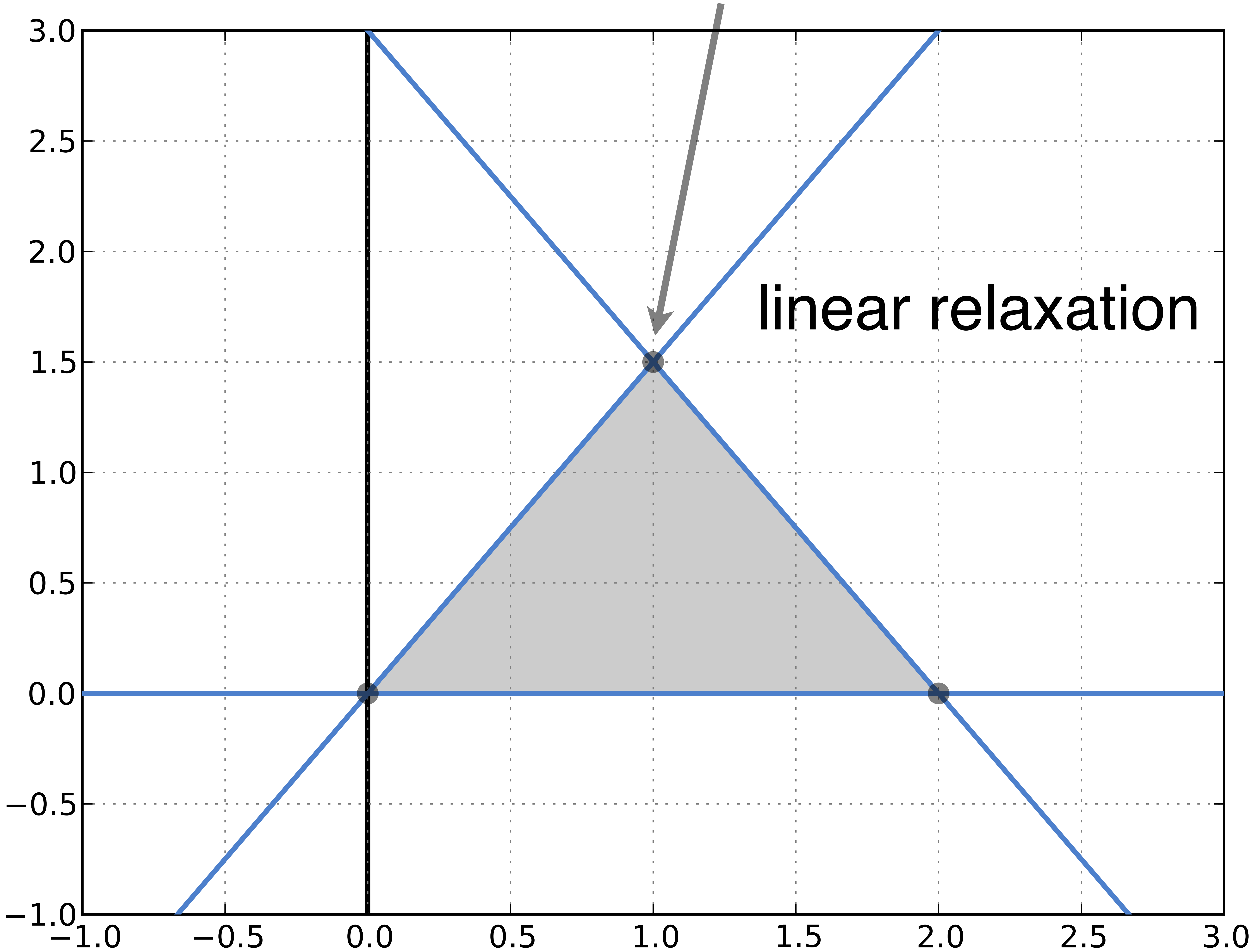

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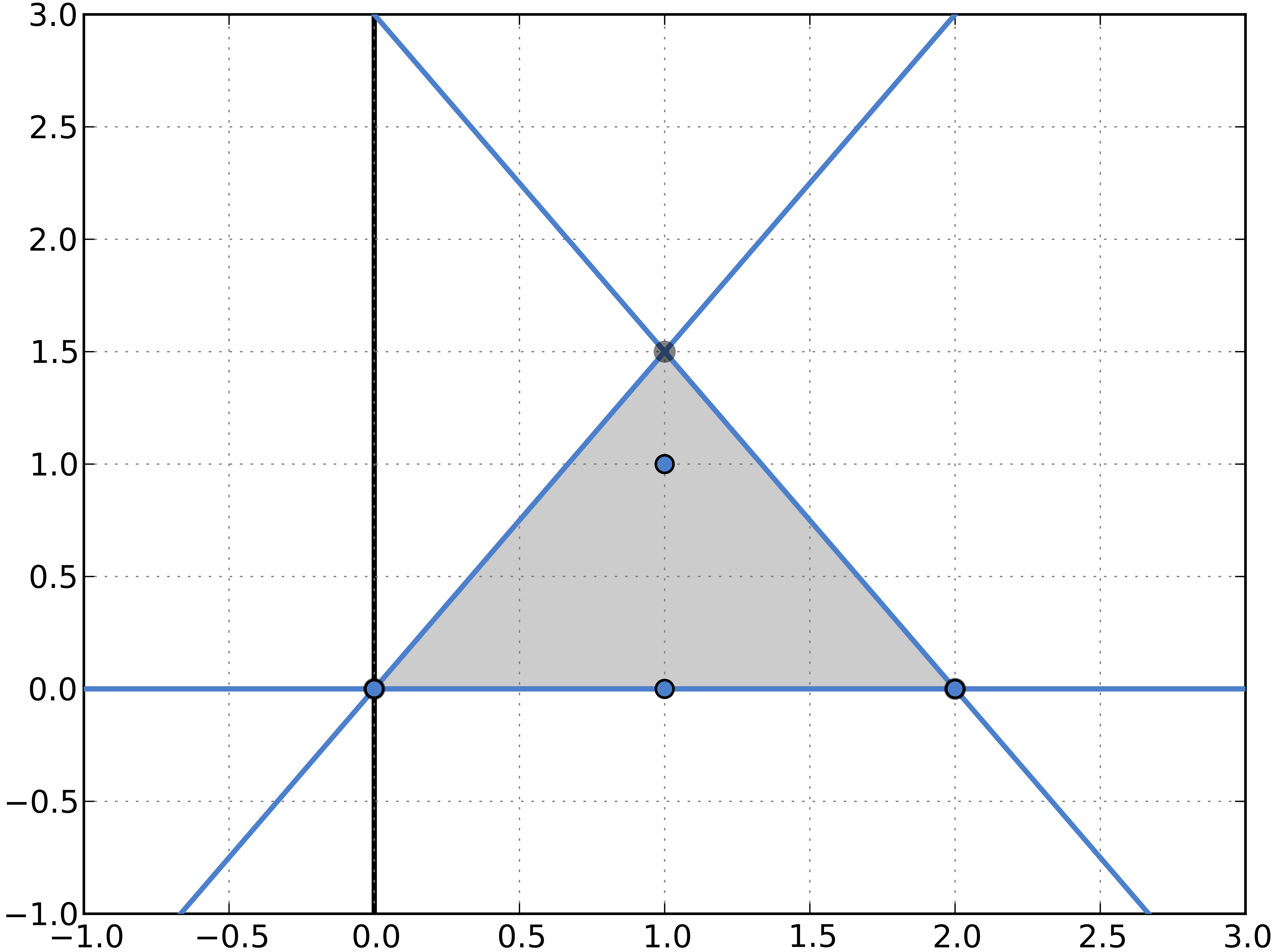

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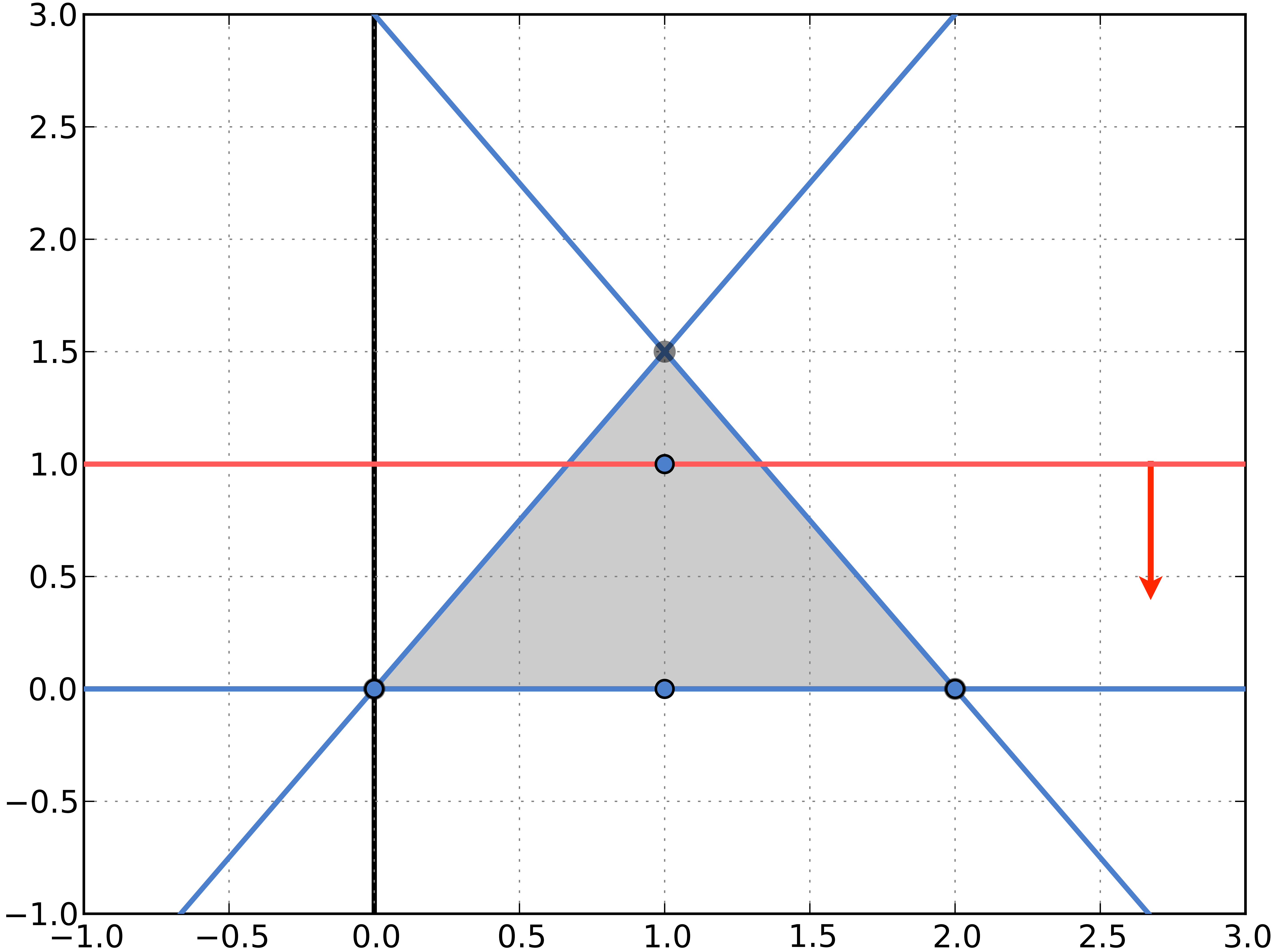

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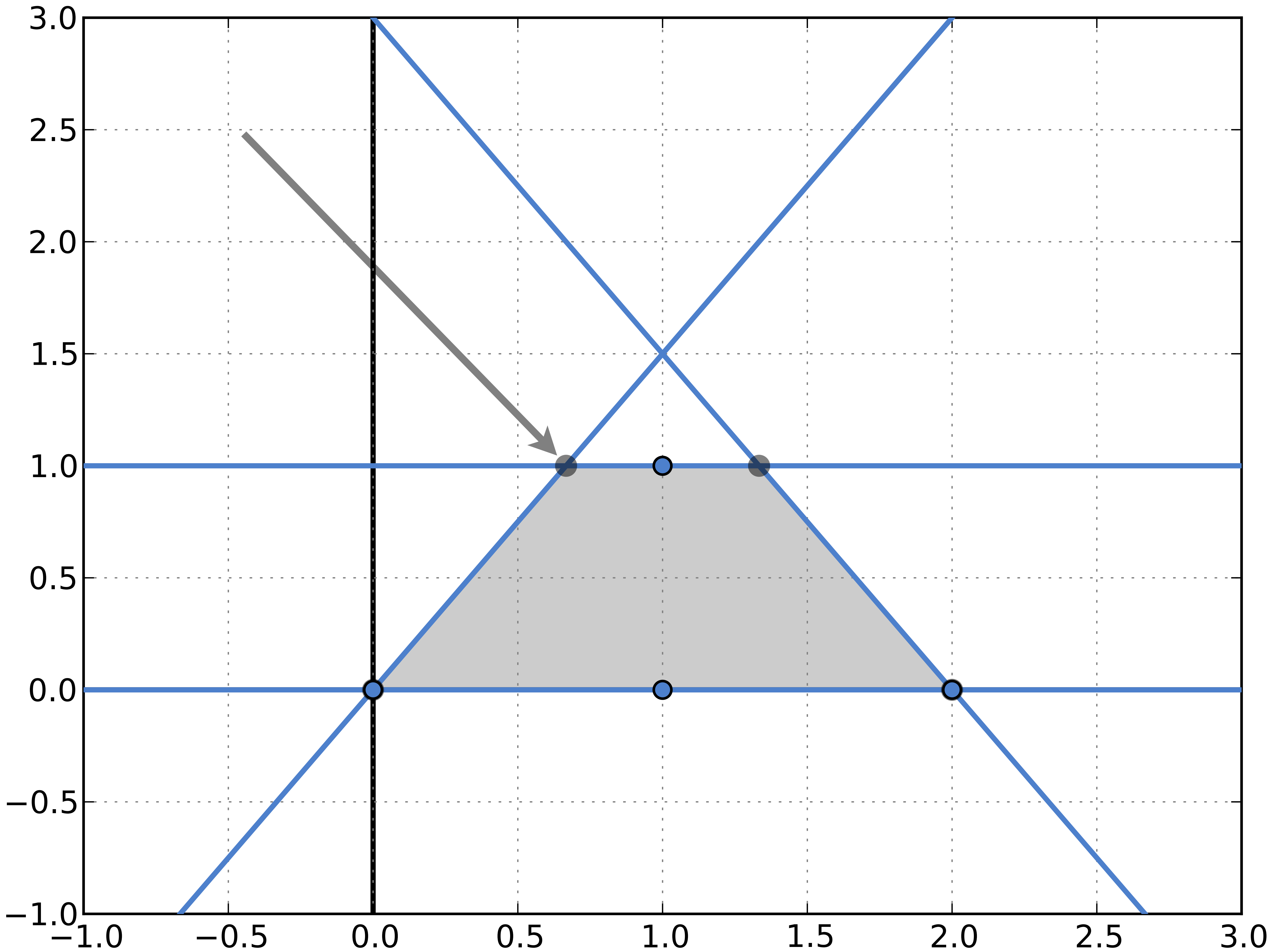

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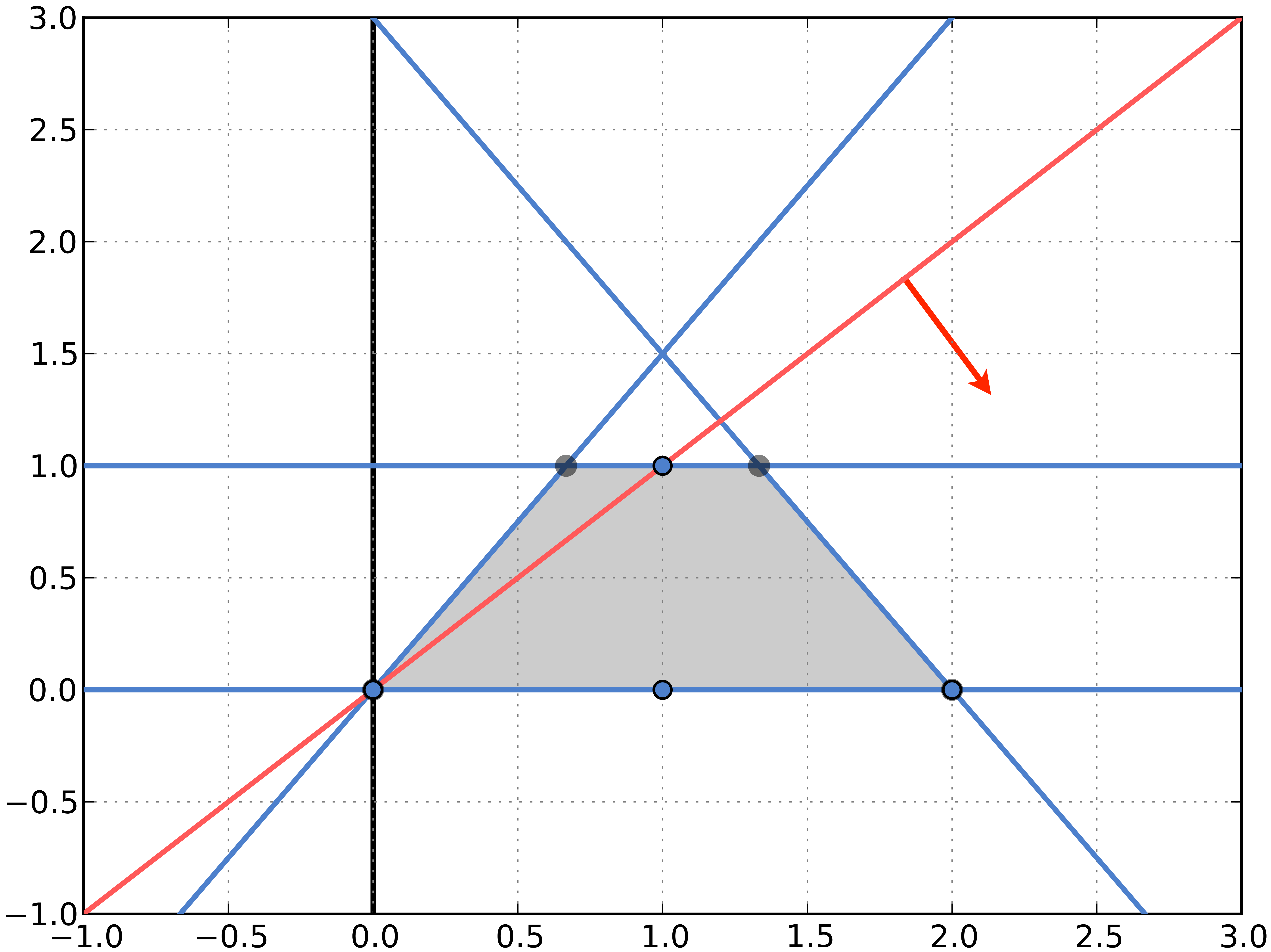

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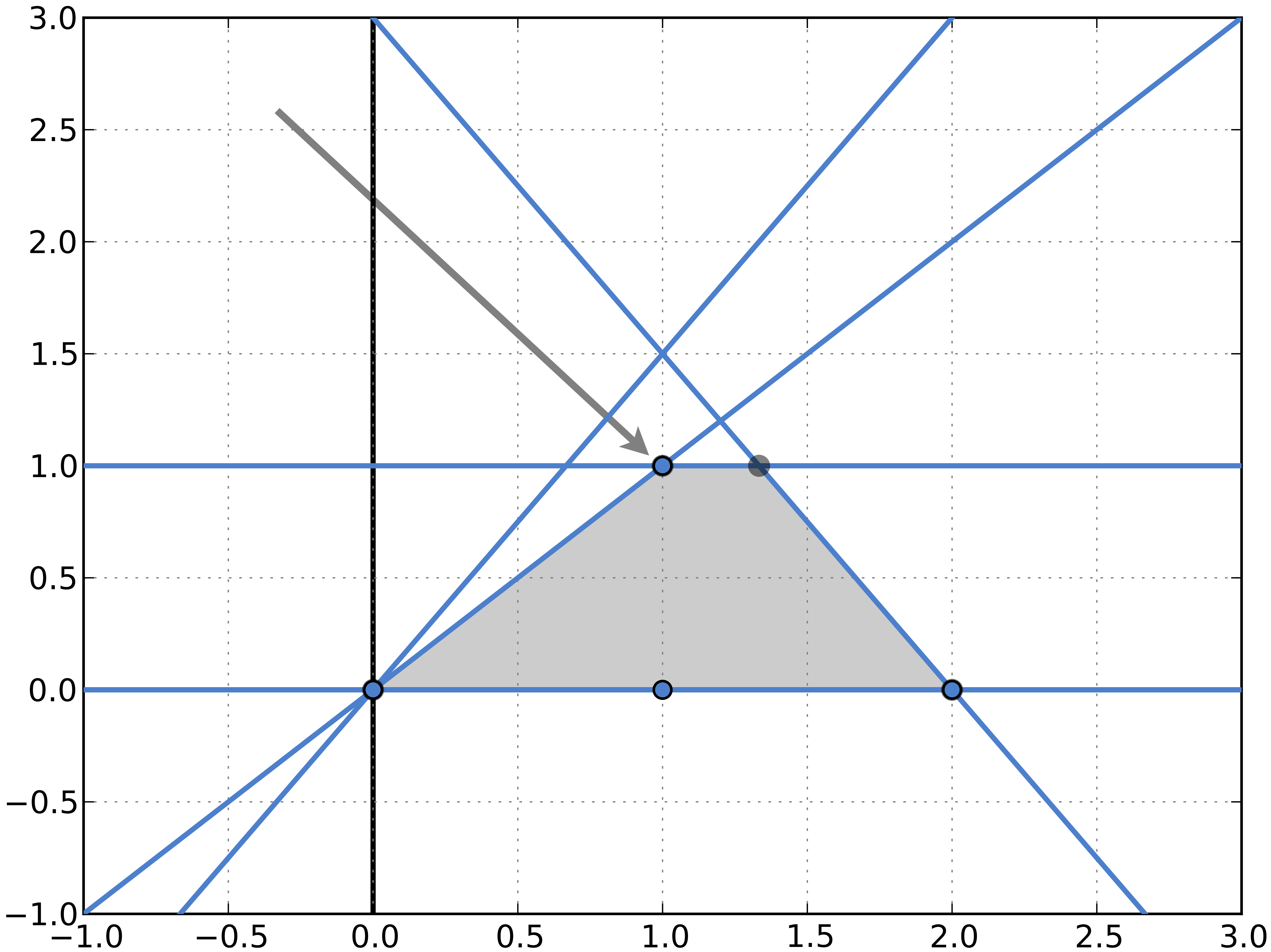

How do we find these cuts?

- Today's lecture: look inside the tableau
- Assumptions
  - all variables take integer values

Basic Feasible Solutions

$$
\begin{aligned}
x_1 &= b_1 + \sum_{j=m+1} a_{1j} x_j \\
&\ldots \\
x_m &= b_m + \sum_{j=m+1} a_{mj} x_j
\end{aligned}
$$

- Basic feasible solution

$$
\begin{aligned}
x_1 &= b_1 \\
&\ldots \\
x_m &= b_m \\
x_j &= 0 \qquad (m < j \leq n)
\end{aligned}
$$

The Tableau

$$
\begin{aligned}
x_1 \quad &+ \quad \sum_{j=m+1} a_{1j} x_j \quad &= \quad b_1 \\
\ldots \\
x_m \quad &+ \quad \sum_{j=m+1} a_{mj} x_j \quad &= \quad b_m
\end{aligned}
$$

- Assume that b_1 is fractional

Gomory Cut

$$x_1 \quad + \quad \sum_{j=m+1} a_{1j} x_j \quad = \quad b_1$$

Gomory Cut

$$x_1 \quad + \sum_{j=m+1} a_{1j}x_j \quad = \quad b_1$$

since $\displaystyle\sum_{j=m+1} \lfloor a_{1j} \rfloor x_j \quad \leq \quad \sum_{j=m+1} a_{1j}x_j$

Gomory Cut

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$$x_1 \quad + \sum_{j=m+1} \lfloor a_{1j} \rfloor x_j \quad \leq \quad b_1$$

Gomory Cut

$$x_1 \quad + \sum_{j=m+1} a_{1j} x_j \quad = \quad b_1$$

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$$x_1 \quad + \sum_{j=m+1} \lfloor a_{1j} \rfloor x_j \quad \leq \quad b_1$$

since $\displaystyle x_1 + \sum_{j=m+1} \lfloor a_{1j} \rfloor x_j$ is an integer

Gomory Cut

$$x_1 + \sum_{j=m+1} a_{1j}x_j = b_1$$

since $\sum_{j=m+1} \lfloor a_{1j} \rfloor x_j \leq \sum_{j=m+1} a_{1j}x_j$

$$x_1 + \sum_{j=m+1} \lfloor a_{1j} \rfloor x_j \leq b_1$$

since $x_1 + \sum_{j=m+1} \lfloor a_{1j} \rfloor x_j$ is an integer

$$x_1 + \sum_{j=m+1} \lfloor a_{1j} \rfloor x_j \leq \lfloor b_1 \rfloor$$

Gomory Cut

$$x_1 + \sum_{j=m+1} \lfloor a_{1j} \rfloor x_j \quad \leq \quad \lfloor b_1 \rfloor$$

- This cut is valid
  - this constraint does not remove any feasible solution

Gomory Cut

$$x_1 + \sum_{j=m+1} \lfloor a_{1j} \rfloor x_j \quad \leq \quad \lfloor b_1 \rfloor$$

- This cut is valid
  - this constraint does not remove any feasible solution
- This cut prunes the basic feasible solution
  - the current basic feasible solution violates it

Reformulating the Gomory Cut

- We can rescale the coefficients of this cut

$$x_1 \quad + \quad \sum_{j=m+1} a_{1j} x_j \quad = \quad b_1$$

-

$$x_1 + \sum_{j=m+1} \lfloor a_{1j} \rfloor x_j \quad \leq \quad \lfloor b_1 \rfloor$$

=

$$\sum_{j=m+1} (a_{1j} - \lfloor a_{1j} \rfloor) x_j \quad \geq \quad b_1 - \lfloor b_1 \rfloor$$

Reformulating the Gomory Cut

$$\sum_{j=m+1} (a_{1j} - \lfloor a_{1j} \rfloor) x_j \quad \geq \quad b_1 - \lfloor b_1 \rfloor$$

- What do we do with this cut?

Reformulating the Gomory Cut

$$\sum_{j=m+1} (a_{1j} - \lfloor a_{1j} \rfloor) x_j \quad \geq \quad b_1 - \lfloor b_1 \rfloor$$

- What do we do with this cut?

$$\sum_{j=m+1} (a_{1j} - \lfloor a_{1j} \rfloor) x_j \quad - \quad s \quad = \quad b_1 - \lfloor b_1 \rfloor$$

Reformulating the Gomory Cut

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$$\sum_{j=m+1} (a_{1j} - \lfloor a_{1j} \rfloor) x_j \quad - \quad s \quad = \quad b_1 - \lfloor b_1 \rfloor$$

$$s = -(b_1 - \lfloor b_1 \rfloor) + \sum_{j=m+1} (a_{1j} - \lfloor a_{1j} \rfloor) x_j$$

Reformulating the Gomory Cut

$$\sum_{j=m+1} (a_{1j} - \lfloor a_{1j} \rfloor)x_j \quad \geq \quad b_1 - \lfloor b_1 \rfloor$$

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$$s = -(b_1 - \lfloor b_1 \rfloor) + \sum_{j=m+1} (a_{1j} - \lfloor a_{1j} \rfloor)x_j$$

- Obviously primal infeasible but dual feasible

Adding Gomory cuts

- Solve the linear relaxation
- Choose a row i whose constant is fractional and add the Gomory cut
- Apply the dual simplex to obtain feasibility
- Iterate until
  - the solution is integral; or
  - there is no feasible solution

Illustration

$$
\begin{array}{llllll}
\max & x_2 & & & & \\
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& & & x_i & \geq & 0 \\
& & & x_i & & \textit{integer}
\end{array}
$$

$$
\begin{array}{lllllllllll}
\min & -x_2 & & & & & & & & & \\
\text{subject to} & & & & & & & & & & \\
& 3x_1 & + & 2x_2 & + & x_3 & & & = & 6 & \\
& -3x_1 & + & 2x_2 & & & + & x_4 & = & 0 & \\
& x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0 & \textit{integer}
\end{array}
$$

Illustration

$$
\begin{array}{llllll}
\max & x_2 & & & & \\
\text{subject to} & & & & & \\
& 3x_1 & + & 2x_2 & \leq & 6 \\
& -3x_1 & + & 2x_2 & \leq & 0 \\
& & & x_i & \geq & 0 \\
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\end{array}
$$

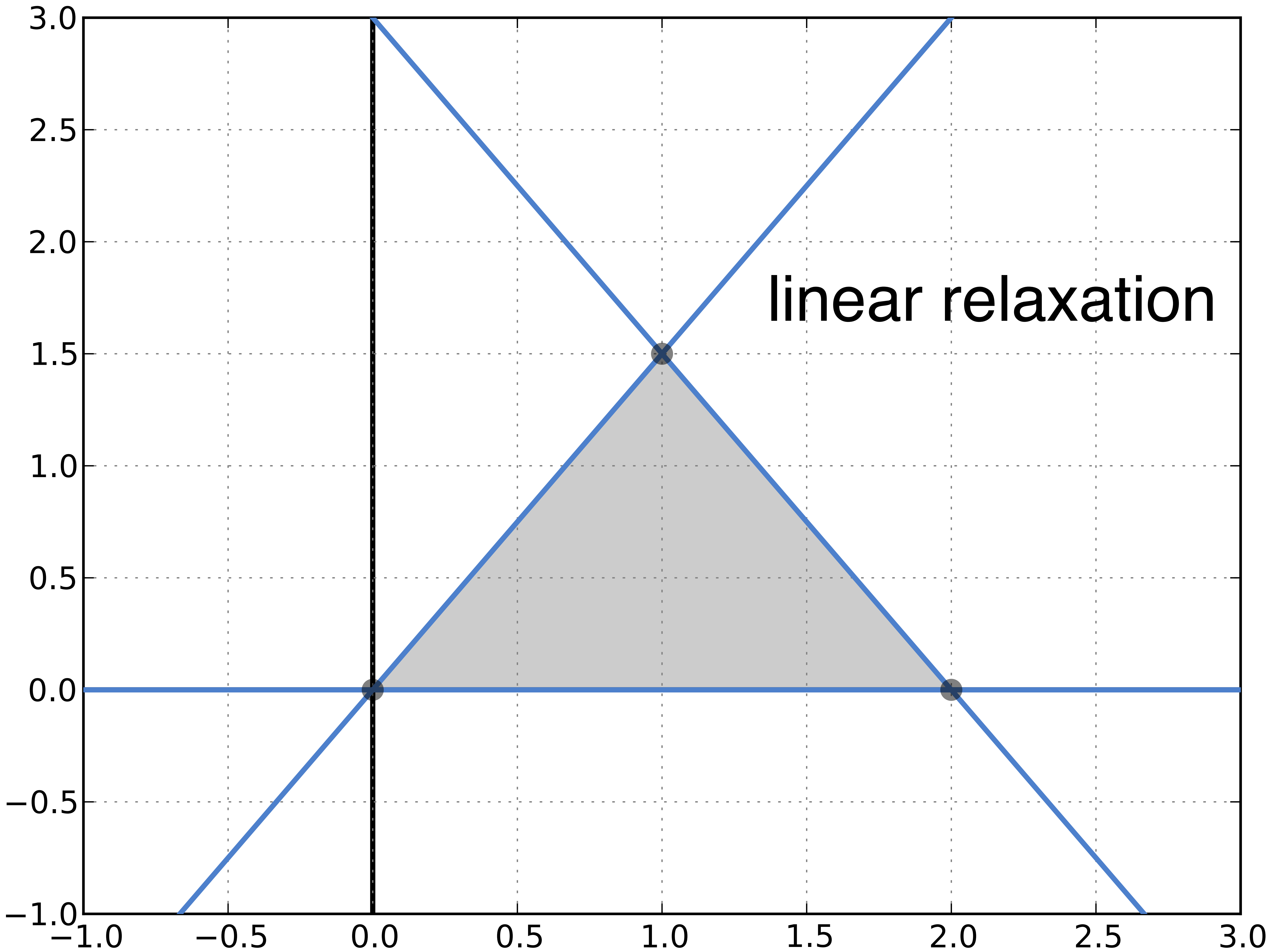
$$\downarrow$$

$$
\begin{array}{llllllllllll}
\min & -x_2 & & & & & & & & & & \\
\text{subject to} & & & & & & & & & & & \\
& 3x_1 & + & 2x_2 & + & x_3 & & & = & 6 & \\
& -3x_1 & + & 2x_2 & & & + & x_4 & = & 0 & \\
& x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0 & \textit{integer}
\end{array}
$$

Illustration

	x_1	x_2	x_3	x_4	b
	0	-1	0	0	0
x_3	3	2	1	0	6
x_4	-3	2	0	1	0

Cutting Planes at Work (geometrically)



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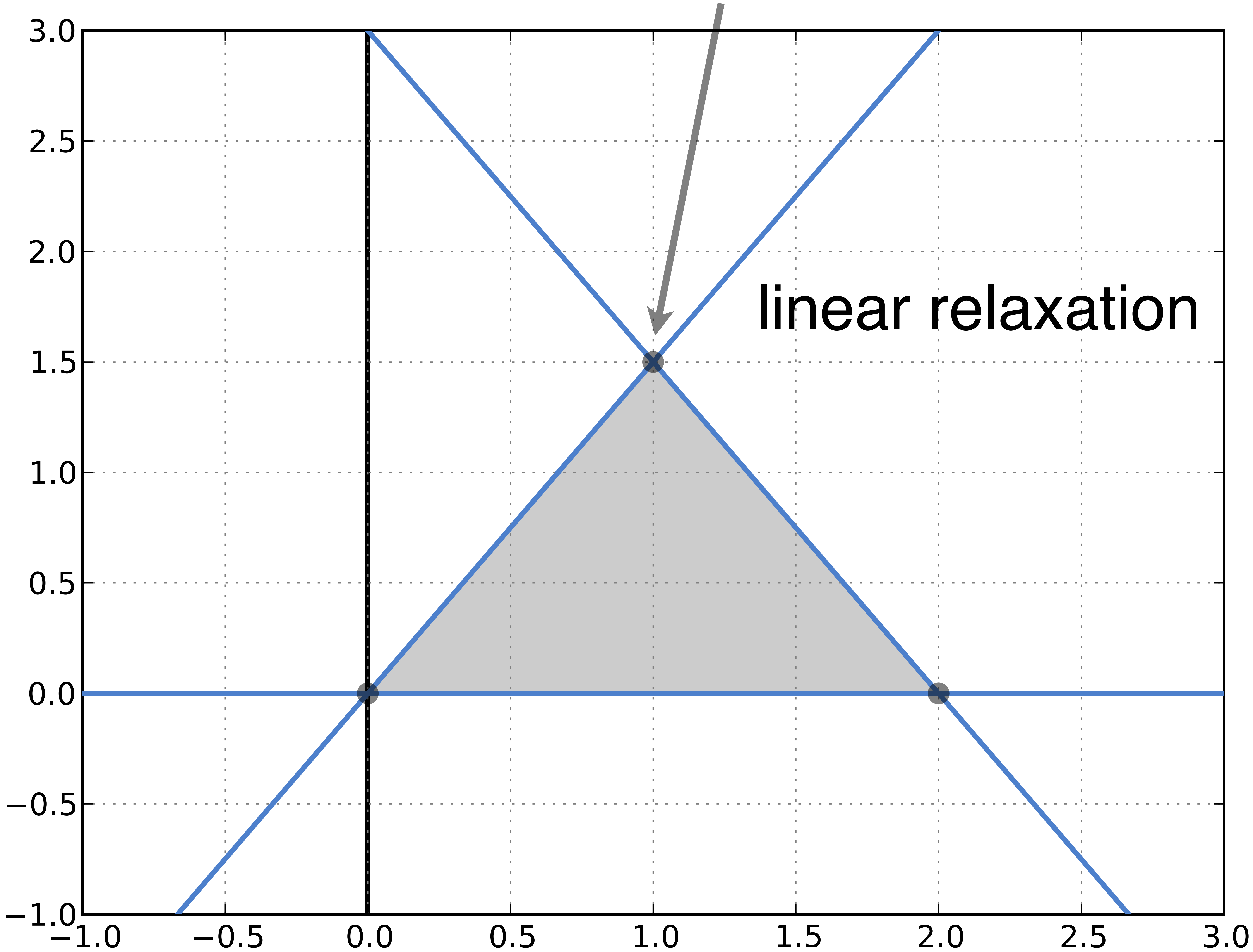


First Gomory Cut

	x_1	x_2	x_3	x_4	b
	0	0	1/4	1/4	3/2
x_1	1	0	1/6	-1/6	1
x_2	0	1	1/4	1/4	3/2

First Gomory Cut

	x_1	x_2	x_3	x_4	b
	0	0	1/4	1/4	3/2
x_1	1	0	1/6	-1/6	1
x_2	0	1	1/4	1/4	3/2

First Gomory Cut

	x_1	x_2	x_3	x_4	b
	0	0	1/4	1/4	3/2
x_1	1	0	1/6	-1/6	1
x_2	0	1	1/4	1/4	3/2

$$\frac{1}{4}x_3 + \frac{1}{4}x_4 \geq \frac{1}{2}$$

What Does this Cut Correspond To?

$$\frac{1}{4}x_3 + \frac{1}{4}x_4 \geq \frac{1}{2}$$

- Re-express in terms of x_1 and x_2

$$\begin{array}{lllllllll}
\min & -x_2 \\
\text{subject to} \\
& 3x_1 & + & 2x_2 & + & x_3 & & & = & 6 \\
& -3x_1 & + & 2x_2 & & & + & x_4 & = & 0 \\
& x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0 \quad \textit{integer}
\end{array}$$

What Does this Cut Correspond To?

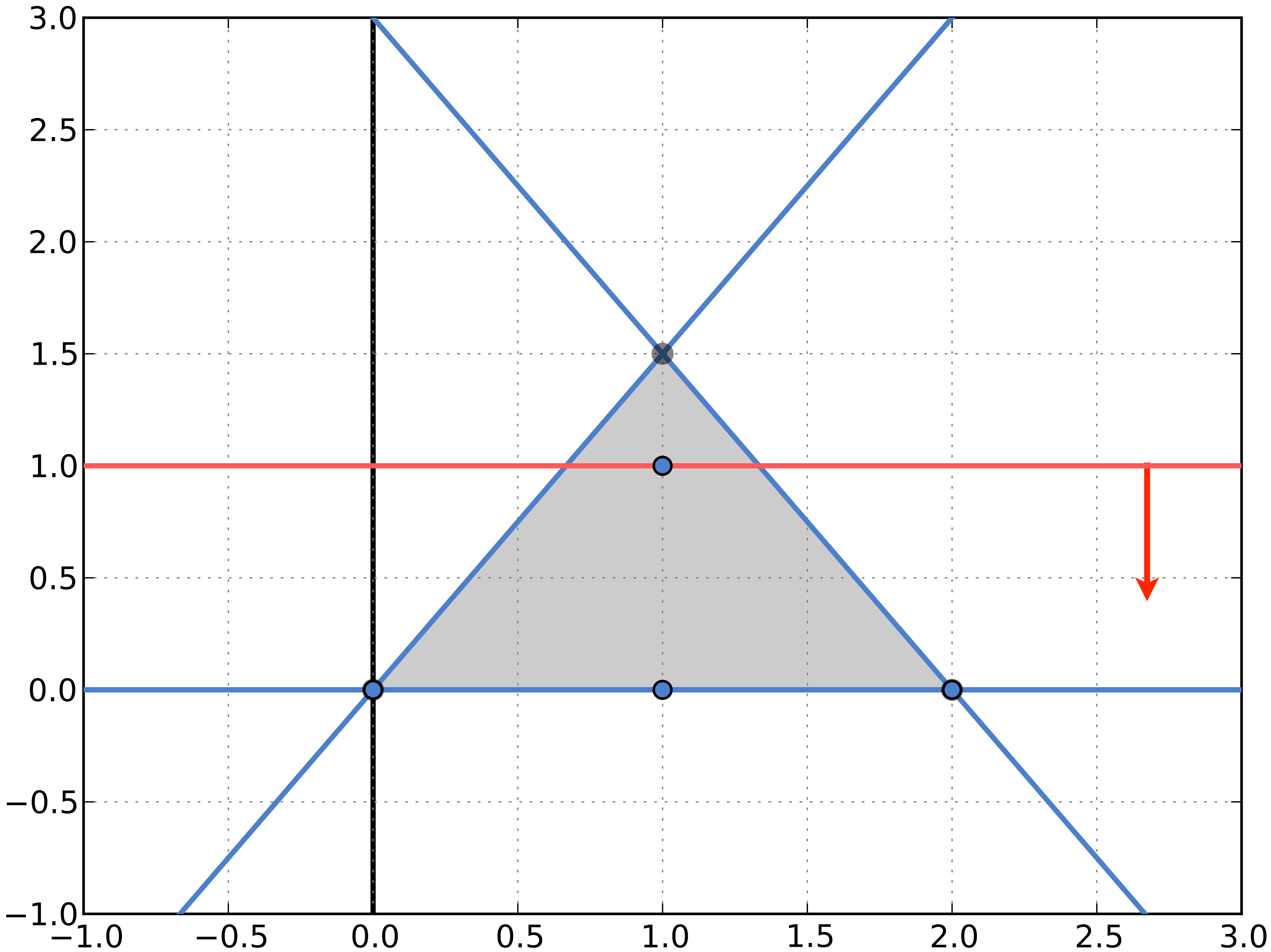
$$\frac{1}{4}x_3 + \frac{1}{4}x_4 \geq \frac{1}{2}$$

- Re-express in terms of x_1 and x_2

$$\frac{1}{4}(6 - 3x_1 - 2x_2) + \frac{1}{4}(3x_1 - 2x_2) \geq \frac{1}{2}$$

$$x_2 \leq 1$$

Cutting Planes at Work (geometrically)



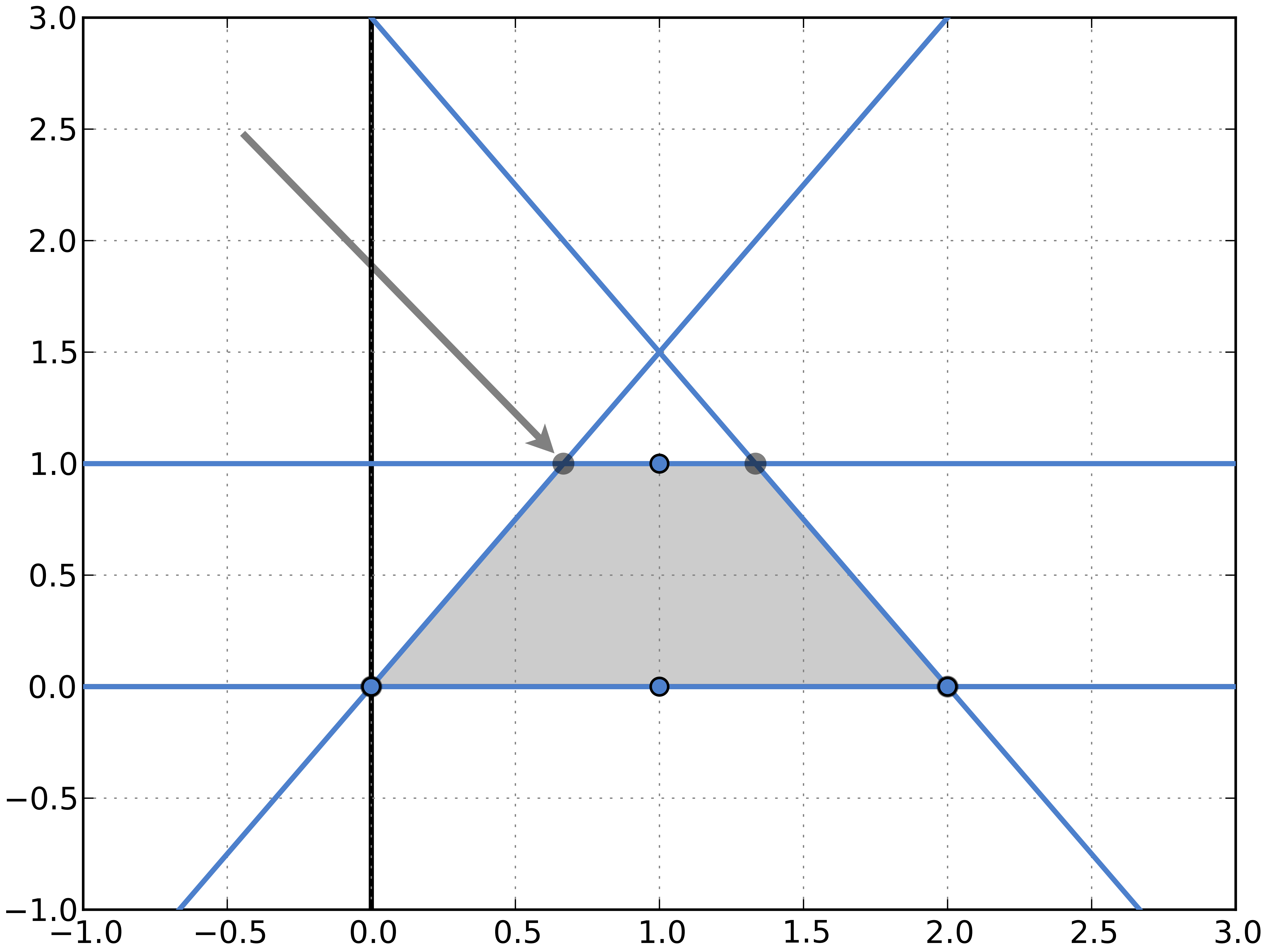
Illustration

	x_1	x_2	x_3	x_4	s_1	b
	0	0	1/4	1/4	0	3/2
x_1	1	0	1/6	-1/6	0	1
x_2	0	1	1/4	1/4	0	3/2
s_1	0	0	-1/4	-1/4	1	-1/2

Illustration

	x_1	x_2	x_3	x_4	s_1	b
	0	0	1/4	1/4	0	3/2
x_1	1	0	1/6	-1/6	0	1
x_2	0	1	1/4	1/4	0	3/2
s_1	0	0	-1/4	-1/4	1	-1/2

Cutting Planes at Work (geometrically)



Illustration

	x_1	x_2	x_3	x_4	s_1	b
	0	0	0	0	0	1
x_1	1	0	0	-1/3	2/3	2/3
x_2	0	1	0	0	1	1
s_1	0	0	1	1	-4	2

Illustration

	x_1	x_2	x_3	x_4	s_1	b
	0	0	0	0	0	1
x_1	1	0	0	-1/3	2/3	2/3
x_2	0	1	0	0	1	1
s_1	0	0	1	1	-4	2

Illustration

	x_1	x_2	x_3	x_4	s_1	b
	0	0	0	0	0	1
x_1	1	0	0	-1/3	2/3	2/3
x_2	0	1	0	0	1	1
s_1	0	0	1	1	-4	2

$$\frac{2}{3}x_4 + \frac{2}{3}s_1 \geq \frac{2}{3}$$

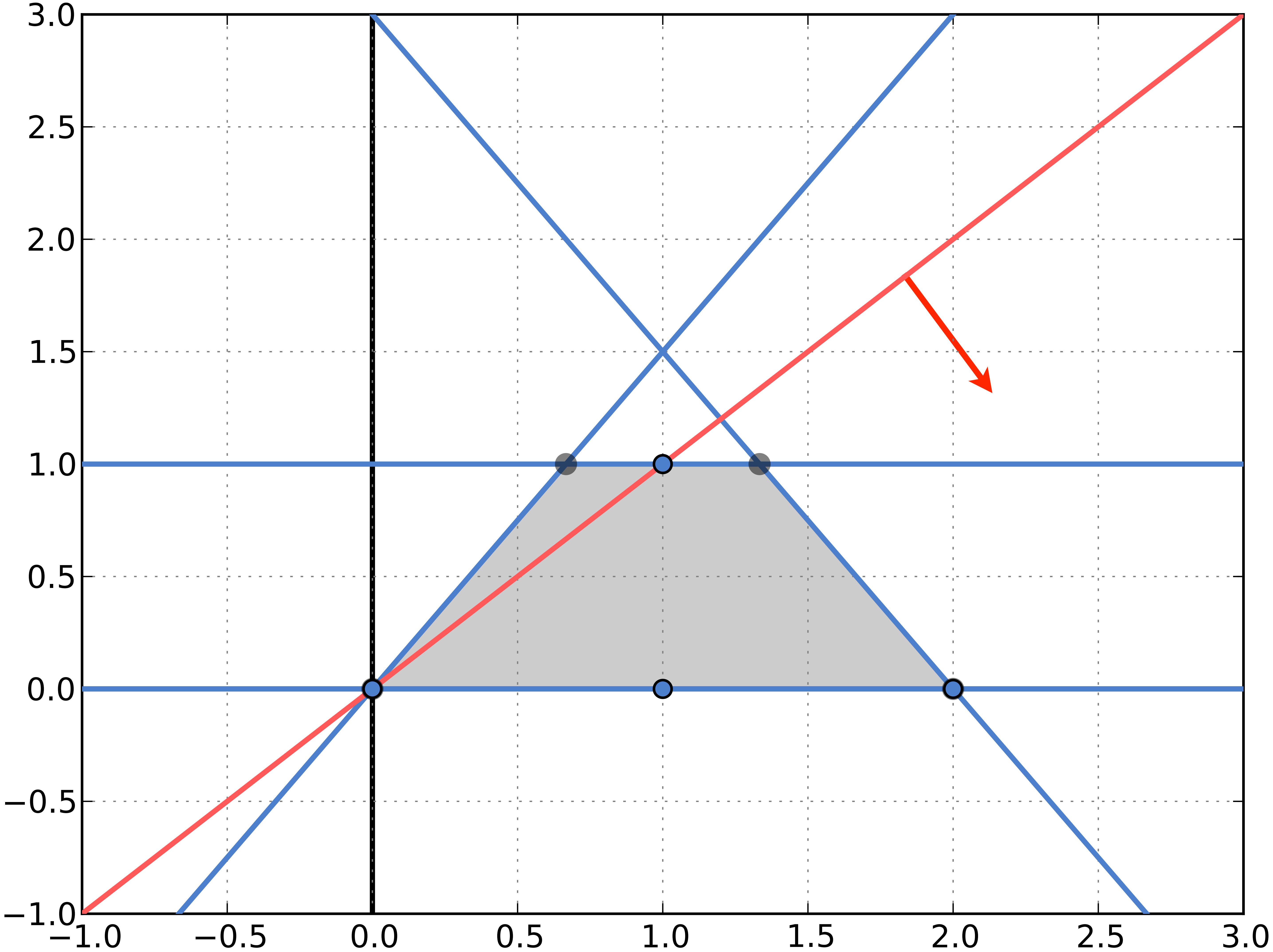
What Does this Cut Correspond To?

$$\frac{2}{3}x_4 + \frac{2}{3}s_1 \geq \frac{2}{3}$$

- Re-express in terms of x_1 and x_2

$$x_1 - x_2 \geq 0$$

Cutting Planes at Work (geometrically)



Gomory Cuts (1958)

Ralph Gomory

Mixed Integer Versus Linear Programs?

- Solve the linear relaxation
- Choose a row i whose constant is fractional and add the Gomory cut
- Apply the dual simplex to obtain feasibility
- iterate until
  - the solution is integral; or
  - there is no feasible solution

Mixed Integer Versus Linear Programs?

- Integrality constraints
  – the gap between P and NP

- Solve the linear relaxation
- Choose a row i whose constant is fractional and add the Gomory cut
- Apply the dual simplex to obtain feasibility
- iterate until

– the solution is integral; or

– there is no feasible solution

The Revival of Gomory Cuts

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- Balas, Ceria, Cornuejols, Natraj (1994-1996)
  - demonstrated how to integrate Gomory cuts in branch and bound
  - generated multiple cuts at once
  - generate them as long as they are "useful"
    - "significant" improvement in the objective value

The Revival of Gomory Cuts

- Balas, Ceria, Cornuejols, Natraj (1994-1996)
  - demonstrated how to integrate Gomory cuts in branch and bound
  - generated multiple cuts at once
  - generate them as long as they are “useful”
    - “significant” improvement in the objective value
- Gomory cuts are now used in state-of-the-art MIP systems

Until Next Time