Discrete Optimization

Mixed Integer Programming: Part II

Goals of the Lecture

- Modeling techniques for MIP models
  - what is a good model?
  - some basic modeling techniques

Branch and Bound for MIP models

- When is Branch and Bound effective?
  - need to prune suboptimal solutions early
  - necessary condition: the linear relaxation is strong
    - is it sufficient?

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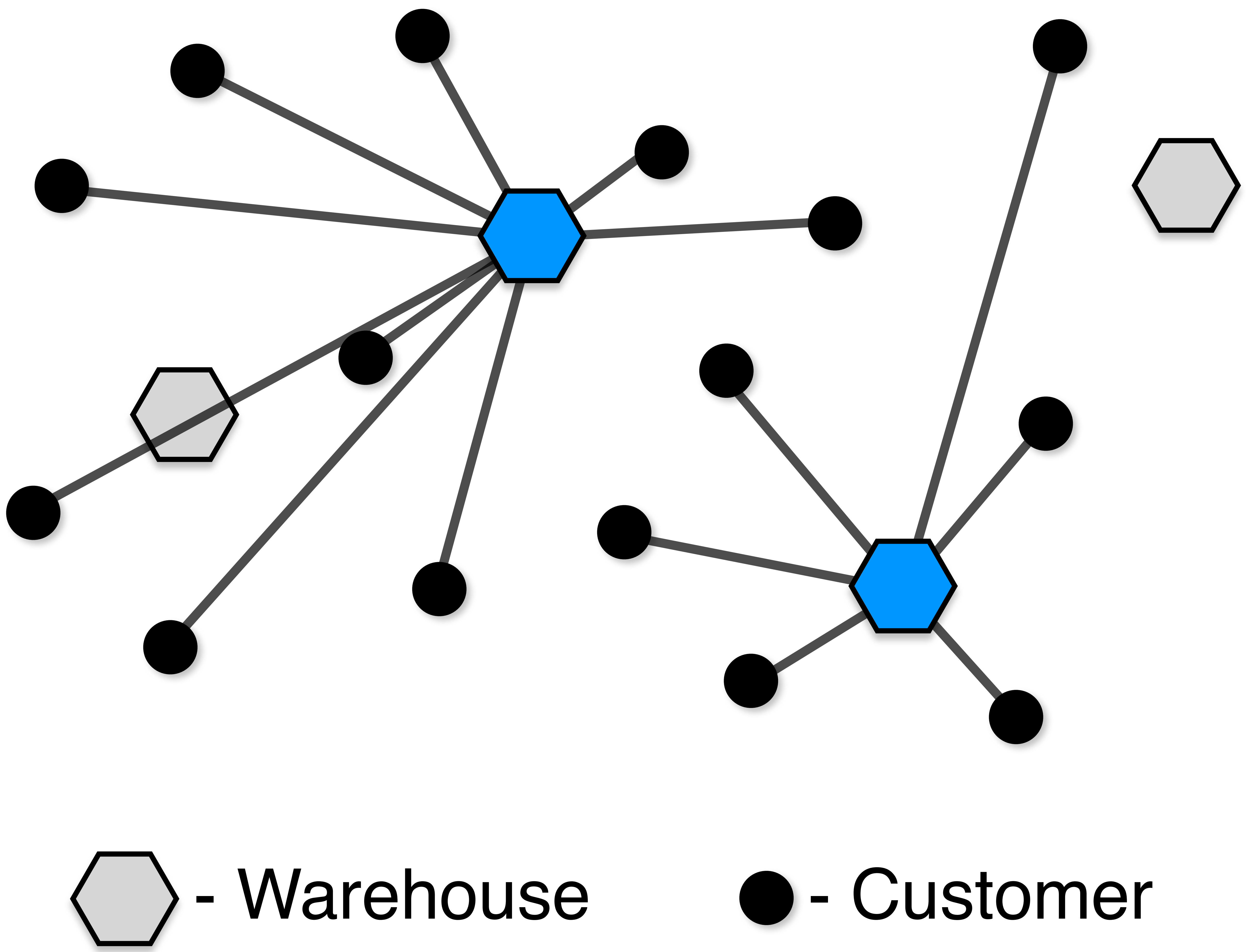
- What is a good MIP model?
  - one with a good linear relaxation

Warehouse Location



Warehouse Location

- Decision variables
  - for each warehouse, decide whether to open it
    - $x_w = 1$ if warehouse w is open
  - decide whether a warehouse serves a customer
    - $y_{wc} = 1$ if warehouse w serves customer c
- What are the constraints?
  - a warehouse can serve a customer only if it is open
  - a customer must be served by exactly one warehouse

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$$y_{w,c} \leq x_w$$

  - a customer must be served by exactly one warehouse

$$\sum_{w \in W} y_{w,c} = 1$$

Warehouse Location

$$
\begin{aligned}
\min \quad & \sum_{w \in W} c_w \; x_w + \sum_{w \in W, c \in C} t_{w,c} \; y_{w,c} \\
\text{subject to} \quad & \\
& y_{w,c} \leq x_w & (w \in W, c \in C) \\
& \sum_{w \in W} y_{w,c} = 1 & (c \in C) \\
& x_w \in \{0,1\} & (w \in W) \\
& y_{w,c} \in \{0,1\} & (w \in W, c \in C)
\end{aligned}
$$

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\begin{array}{lll}
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- Can we express this constraint differently?

$$\sum_{c \in C} y_{wc} \leq |C| x_w$$

Warehouse Location

- The new model

$$\min \quad \sum_{w \in W} c_w \; x_w + \sum_{w \in W, c \in C} t_{w,c} \; y_{w,c}$$

subject to

$$\sum_{c \in C} y_{w,c} \leq |C| x_w \qquad (w \in W)$$

$$\sum_{w \in W} y_{w,c} = 1 \qquad (c \in C)$$

$$x_w \in \{0, 1\} \qquad (w \in W)$$

$$y_{w,c} \in \{0, 1\} \qquad (w \in W, c \in C)$$

Warehouse Location

- Which of the two models is best?
  - our new model has a single constraint instead of |C| constraints for each warehouse

Warehouse Location

- Which of the two models is best?
  - our new model has a single constraint instead of |C| constraints for each warehouse

- What about the quality of linear relaxation?

Warehouse Location

$$y_{w,c} \leq x_w \quad (w \in W, c \in C)$$

Warehouse Location

- A solution to

$$y_{w,c} \leq x_w \quad (w \in W, c \in C)$$

Warehouse Location

- A solution to

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- but not vice-versa.

- So the initial model has a better linear relaxation!

Warehouse Location Relaxations

$$y_{w,c} \leq x_w \quad (w \in W, c \in C)$$

W	C	OBJ_1	OBJ_2	%
16	50	932,615	844,807	9.5
16	50	1,010,641	853,434	15.6
25	50	796,648	659,341	17.2
50	50	793,439	631,421	20.4

$$\sum_{c \in C} y_{w,c} \leq |C| x_w \quad (w \in W)$$

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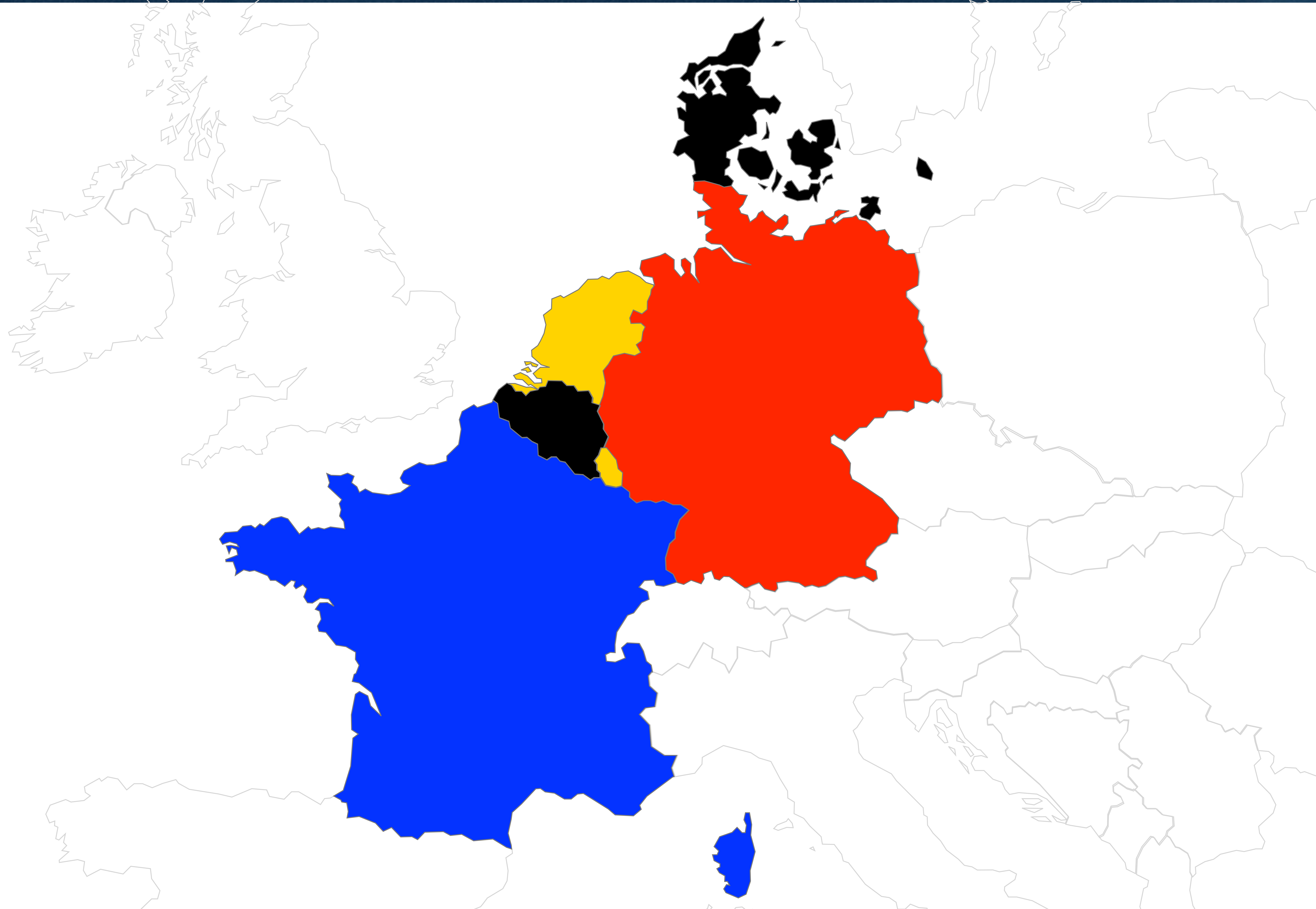
Coloring a Map



The Coloring Problem

```
enum Countries = { Belgium, Denmark, France,
                   Germany, Netherlands, Luxembourg };
var{int} color[Countries] in 0..3;
minimize
 max(c in Countries) color[c]
subject to {
 color[Belgium] ≠ color[France];
 color[Belgium] ≠ color[Germany];
 color[Belgium] ≠ color[Netherlands];
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 color[Denmark] ≠ color[Germany];
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Big-M Transformation

$$x \neq y$$

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- The disjunction is not allowed in a MIP model
- Introduce a 0/1 variable ***b*** and a large number ***M***

$$x \leq y - 1 + b\ M$$
$$x \geq y + 1 - (1 - b)M$$

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- Introduce a 0/1 variable ***b*** and a large number ***M***

$$\begin{aligned} x &\leq y - 1 + b\ M \\ x &\geq y + 1 - (1 - b)M \end{aligned}$$

- This is the big-M rewriting of $x \neq y$

Big-M Transformation

$$x \leq y - 1 + b\ M$$
$$x \geq y + 1 - (1 - b)M$$

Big-M Transformation

- What is the intuition?

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- Constraint $x \leq y - 1 + b\ M$ is trivially satisfied when b = 1. The second constraint then becomes

$$x \geq y + 1$$

Big-M Transformation

- What is the intuition?

$$x \leq y - 1 + b\ M$$
$$x \geq y + 1 - (1 - b)M$$

- Constraint $x \leq y - 1 + b\ M$ is trivially satisfied when b = 1. The second constraint then becomes

$$x \geq y + 1$$

- Constraint $x \geq y + 1 - (1 - b)M$ is trivially satisfied when b = 0. The first constraint becomes

$$x \leq y - 1$$

Big-M Transformation

- What is the linear relation going to do?

$$x \leq y - 1 + b\ M$$
$$x \geq y + 1 - (1 - b)M$$

Big-M Transformation

- What is the linear relation going to do?

$$x \leq y - 1 + b\ M$$
$$x \geq y + 1 - (1 - b)M$$

- Choose b = 0.5
  - half of a big number is still a big number

Coloring a Map with Big-M

$$
\begin{aligned}
&obj && \in \{0,1,2,3\}\\
&\mathrm{color}_c && \in \{0,1,2,3\}\\
&b_{c1,c2} && \in \{0,1\}\\
&M && = 4
\end{aligned}
$$

$$
\begin{aligned}
&\min && obj\\
&\text{subject to} && \\
& && obj \geq \mathrm{color}_c \quad (c \in C)\\
& && \mathrm{color}_{c_1} \leq \mathrm{color}_{c_2} - 1 + b_{c_1,c_2} M\\
& && \mathrm{color}_{c_1} \geq \mathrm{color}_{c_2} + 1 - (1 - b_{c_1,c_2}) M\\
& && (c_1, c_2 \in C \text{ and adjacent})
\end{aligned}
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Coloring a Map with Big-M

the objective is greater than the largest given color

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Coloring a Map with Big-M

the objective is greater than the largest given color

no two adjacent countries are given the same color

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Coloring a Map with Big-M

LP: $$\begin{aligned} obj &= 0.0 \\ \text{color} &= 0.0 \\ b &= 0.25 \end{aligned}$$

$$\begin{aligned} obj &\in \{0,1,2,3\} \\ \text{color}_c &\in \{0,1,2,3\} \\ b_{c1,c2} &\in \{0,1\} \\ M &= 4 \end{aligned}$$

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Need at least 1 color!...

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MIP: Optimal - 5 nodes
Proof - 65 nodes

The Coloring Problem

- Can we find another model?

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MIP Love Affair with 0/1 Variables

- Can we find another model?
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- Consider a variable x with domain 0..3
  - use four 0/1 variables: $b_{x=0}$, $b_{x=1}$, $b_{x=2}$, $b_{x=3}$
  - $b_{x=i}$ is 1 if variable x = i

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  - $b_{x=i}$ is 1 if variable x = i
- Add the constraint

$$b_{x=0} + b_{x=1} + b_{x=2} + b_{x=3} = 1$$

MIP Love Affair with 0/1 Variables

- Can we find another model?
  - binarize all variables
- Consider a constraint $x \neq y$. It becomes

$$
\begin{aligned}
b_{x=0} + b_{y=0} &\leq 1 \\
b_{x=1} + b_{y=1} &\leq 1 \\
b_{x=2} + b_{y=2} &\leq 1 \\
b_{x=3} + b_{y=3} &\leq 1
\end{aligned}
$$

Coloring a Map with 0/1 Variables

$$
\begin{array}{ll}
obj & \in \{0, 1, 2, 3\} \\
\text{color}_{c,v} & \in \{0, 1\}
\end{array}
$$

$$
\begin{array}{lll}
\text{min} & obj & \\
\text{subject to} & & \\
& obj \geq v \times \text{color}_{c,v} & (c \in C, v \in 0..3) \\
& \displaystyle\sum_{v=0}^{3} \text{color}_{c,v} = 1 & (c \in C) \\
& \text{color}_{c_1,v} + \text{color}_{c_2,v} \leq 1 & (c_1, c_2 \in C \text{ and adjacent}, v \in 0..3)
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the objective is greater than the largest given color

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a country is given a single color

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no two adjacent countries are given the same color

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\end{aligned}
$$

Coloring a Map with 0/1 Variables

LP:
$$
\begin{aligned}
obj &= 0.\overline{27} \\
\text{color}_{c,0} &= 0.5 \\
\text{color}_{c,1} &= 0.\overline{27} \\
\text{color}_{c,2} &= 0.1\overline{36} \\
\text{color}_{c,3} &= 0.\overline{09}
\end{aligned}
$$

$$
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obj &\in \{0,1,2,3\} \\
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\end{aligned}$$

$$\begin{aligned}
obj &\in \{0, 1, 2, 3\} \\
\text{color}_{c,v} &\in \{0, 1\}
\end{aligned}$$

Need at least 2 colors!

$$\begin{array}{lll}
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\end{array}$$

Coloring a Map with 0/1 Variables

MIP:

Optimal - 12 nodes

Proof - 22 nodes

$$
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$$

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\text{color}_{c,2} &= 0.1\overline{36} \\
\text{color}_{c,3} &= 0.\overline{09}
\end{aligned}
$$

Need at least 2 colors!

Combining Constraints

$$obj \geq v \times \mathrm{color}_{c,v} \quad (c \in C, v \in 0..3)$$

$$\sum_{v=0}^{3} \mathrm{color}_{c,v} = 1 \quad (c \in C)$$

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$$\downarrow$$

$$obj \geq \sum_{v=0}^{3} v \times \text{color}_{c,v} \quad (c \in C)$$

Coloring a Map with 0/1 Variables

$$
\begin{aligned}
&obj && \in \{0,1,2,3\} \\
&\mathrm{color}_{c,v} && \in \{0,1\} \\
&\min && obj \\
&\text{subject to} && obj \geq \sum_{v=0}^{3} v \times \mathrm{color}_{c,v} && (c \in C) \\
& && \sum_{v=0}^{3} \mathrm{color}_{c,v} = 1 && (c \in C) \\
& && \mathrm{color}_{c_1,v} + \mathrm{color}_{c_2,v} \leq 1 && (c_1, c_2 \in C \text{ and adjacent}, v \in 0..3)
\end{aligned}
$$

Coloring a Map with 0/1 Variables

LP:
$$\begin{aligned} obj &= 0.5 \\ \text{color}_{c,0} &= 0.5 \\ \text{color}_{c,1} &= 0.5 \\ \text{color}_{c,2} &= 0 \\ \text{color}_{c,3} &= 0 \end{aligned}$$

$$\begin{aligned} obj &\in \{0,1,2,3\} \\ \text{color}_{c,v} &\in \{0,1\} \end{aligned}$$

$$\begin{array}{lll} \text{min} & obj & \\ \text{subject to} & obj \geq \sum_{v=0}^{3} v \times \text{color}_{c,v} & (c \in C) \\ & \sum_{v=0}^{3} \text{color}_{c,v} = 1 & (c \in C) \\ & \text{color}_{c_1,v} + \text{color}_{c_2,v} \leq 1 & (c_1, c_2 \in C \text{ and adjacent}, v \in 0..3) \end{array}$$

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Need at least 2 colors!

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Need at least 2 colors!

Coloring a Map with 0/1 Variables

MIP:

Optimal - 9 nodes
Proof - 41 nodes

$$\begin{aligned} obj &\in \{0,1,2,3\} \\ \mathrm{color}_{c,v} &\in \{0,1\} \end{aligned}$$

$$\begin{aligned} &\min && obj \\ &\text{subject to} && obj \geq \sum_{v=0}^{3} v \times \mathrm{color}_{c,v} && (c \in C) \\ &&& \sum_{v=0}^{3} \mathrm{color}_{c,v} = 1 && (c \in C) \\ &&& \mathrm{color}_{c_1,v} + \mathrm{color}_{c_2,v} \leq 1 && (c_1, c_2 \in C \text{ and adjacent}, v \in 0..3) \end{aligned}$$

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Need at least 2 colors!

Until Next Time