Discrete Optimization

Linear Programming: Part VI

Goals of the Lecture

- Linear programming
  - where is this dual coming from?
  - what does it mean?
  - what is useful for?

Bounding

$$
\begin{array}{lrcrcrcrcr}
\text{max} & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & \\
\text{subject to} & & & & & & & & & \\
& x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 \\
& 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 \\
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\end{array}
$$

- can we find an upper bound?

$$
10x_1 \quad + \quad 2x_2 \quad + \quad 6x_3 \quad + \quad 16x_4 \quad \leq \quad 110
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$$\leq$$

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Complementarity Slackness

- What do these dual variables mean?

Let x^* and y^* be the optimal solutions to the primal and dual.
The following conditions are necessary and sufficient
for the optimality of x^* and y^*.

$$\sum_{j=1}^{n} a_{ij} x_j^* = b_i \ \vee y_i^* = 0 \quad (1 \leq i \leq m)$$

and

$$\sum_{i=1}^{n} a_{ij} y_i^* = c_j \ \vee x_j^* = 0 \quad (1 \leq j \leq n)$$

Economic Interpretation

$$\max \quad \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \quad \leq \quad b_i \boxed{+\, t_i} \quad (1 \leq i \leq m)$$

Economic Interpretation

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- for some small t_i, this linear program has an optimal solution

$$z^* + \sum_{i=1}^{m} y_i^* \; t_i$$

optimal primal objective   dual solution

Duality in the Tableau

			0	0	0	0					
			1								
				1							
					1						
						1					

$$c_j - c_B A_B^{-1} A_j$$

Duality in the Tableau

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Duality in the Tableau

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- at optimality, the cost become $c_j - c_B A_B^{-1} A_j$

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  - primal always feasible; dual not

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  - The dual is still feasible (I am adding a variable); the primal is not
  - Optimize the dual and the primal becomes feasible at optimality

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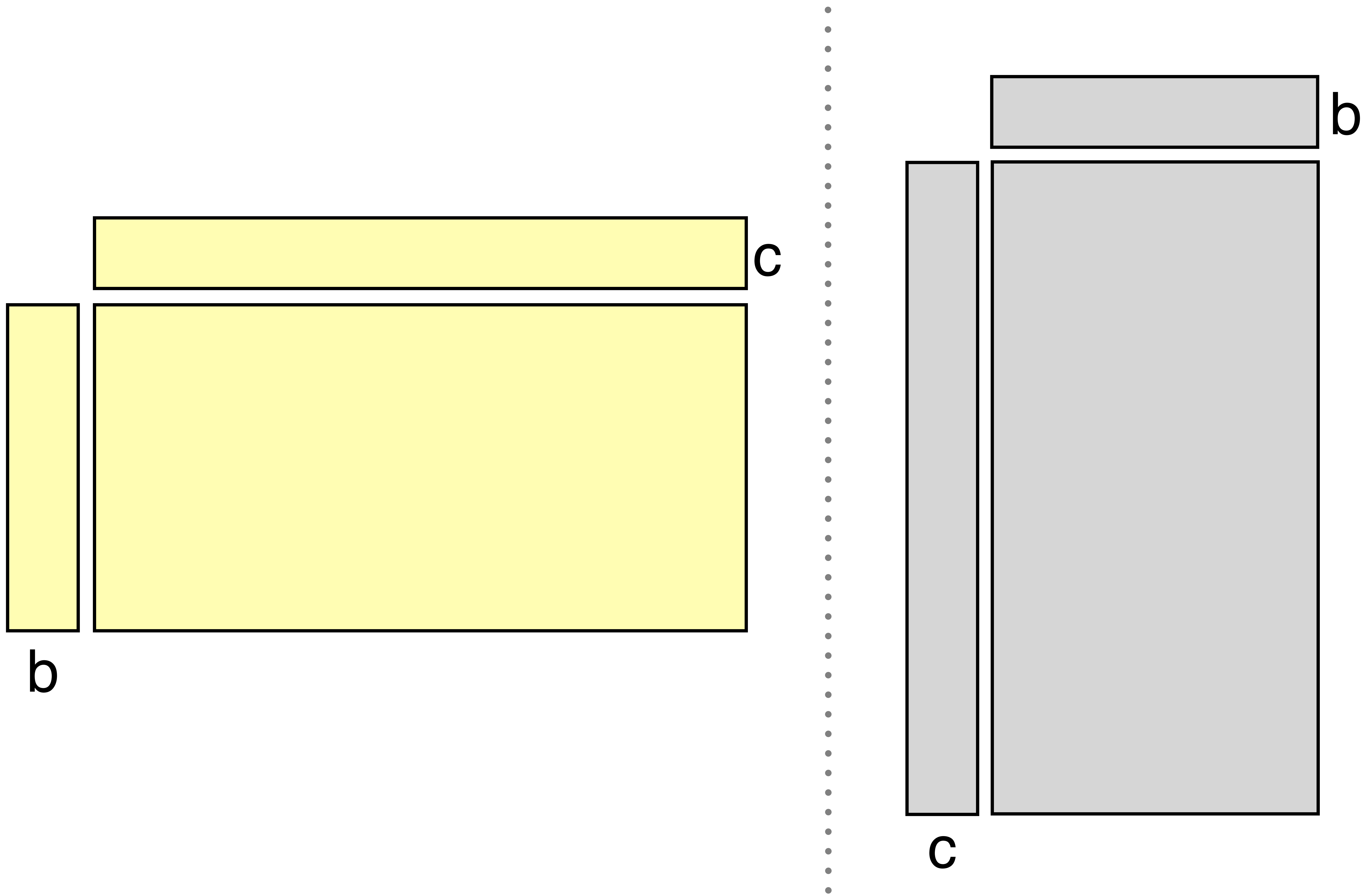
- Use the same tableau

Primal - Dual



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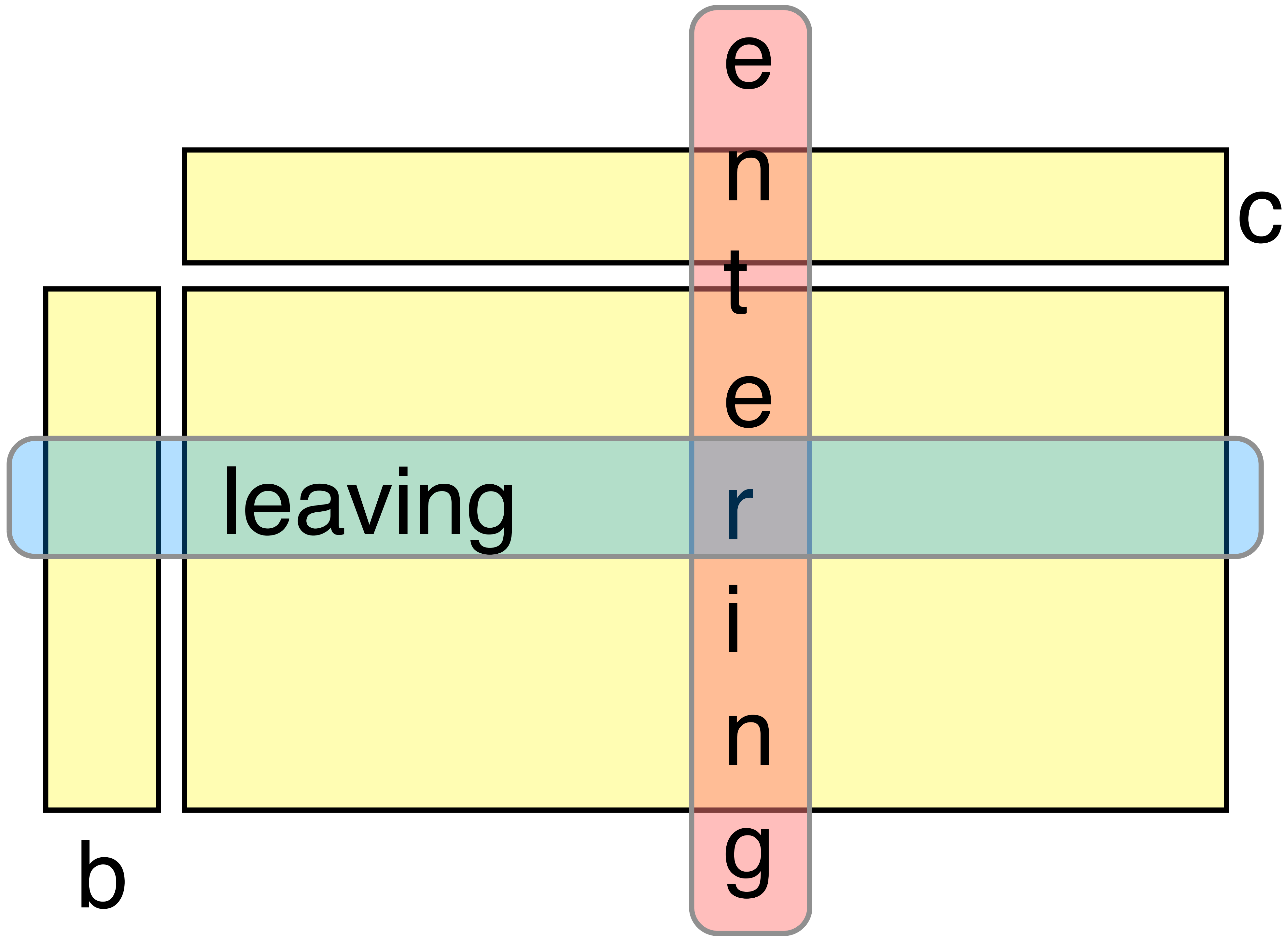

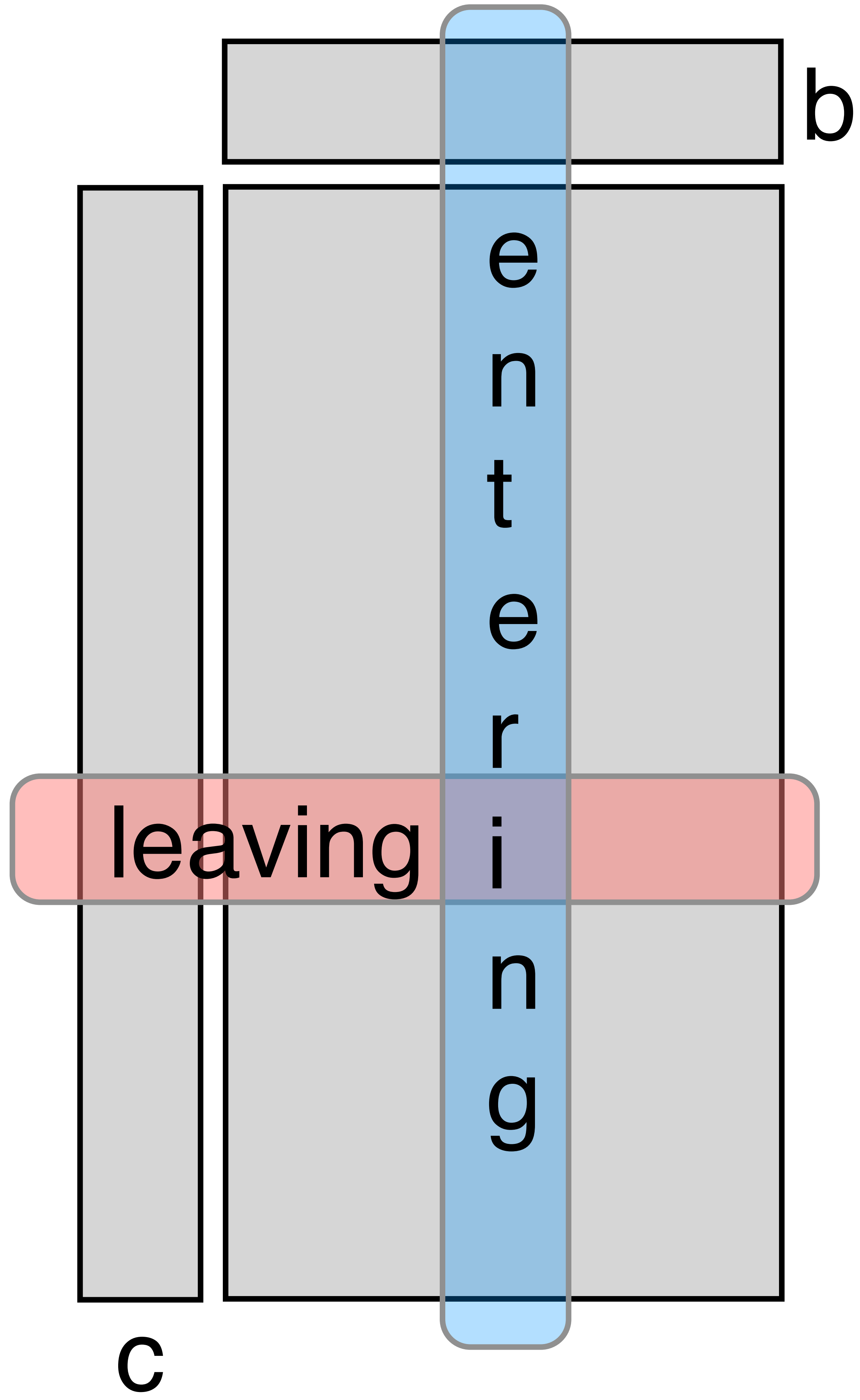

Until Next Time