

Discrete Optimization

Linear Programming: Part V

Duality

min $c^T x$ primal
subject to

$$Ax \geq b$$

$$x_j \geq 0$$

max $y^T b$
subject to

$$yA \leq c$$

$$y_i \geq 0$$

dual

Duality

$$\begin{array}{ll}\min & 3x_1 + 2x_2 + 4x_3 \\ \text{subject to} & \\ & 2x_1 + x_2 \geq 2 \\ & 2x_1 - x_2 + x_3 \geq 5\end{array}$$

$$\begin{array}{ll}\max & 2y_1 + 5y_2 \\ \text{subject to} & \\ & 2y_1 + 2y_2 \leq 3 \\ & y_1 - y_2 \leq 2 \\ & y_2 \leq 4\end{array}$$

How do We Obtain this Dual?

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How do We Obtain this Dual?

$$\begin{array}{llllll} \min & 3x_1 & + & 2x_2 & + & 4x_3 \\ \text{subject to} & 2x_1 & + & x_2 & + & \\ & 2x_1 & - & x_2 & + & x_3 \end{array} \begin{array}{l} \\ \geq 2 \\ \geq 5 \end{array} \begin{array}{l} y_1 \\ y_2 \end{array}$$

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y_1
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 \end{array}
 \quad
 \begin{array}{c}
 y_1 \\
 y_2
 \end{array}$$

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 & \boxed{2y_1} & + & 2y_2 & \leq 3 \\
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How do We Obtain this Dual?

min
subject to

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$$\begin{array}{cc} 2x_1 & + & x_2 \\ 2x_1 & - & x_2 \end{array} + x_3 \geq \begin{array}{c} 2 \\ 5 \end{array} \quad \begin{array}{c} y_1 \\ y_2 \end{array}$$

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$$\begin{array}{ll}\min & \begin{bmatrix} 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ \text{subject to} & \begin{bmatrix} 2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} 2 \\ 5 \end{bmatrix}\end{array}$$

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Duality

primal

$$\begin{array}{ll}\min & c x \\ \text{subject to} & Ax \geq b \\ & x_j \geq 0\end{array}$$

dual

$$\begin{array}{ll}\max & y b \\ \text{subject to} & yA \leq c \\ & y \geq 0\end{array}$$

- Theorem: If the primal has an optimal solution, the dual has an optimal solution with the same cost

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Let x and Π be feasible solutions to the primal and dual respectively.

We have that $cx \geq \Pi Ax \geq \Pi b$.

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- since the primal has a feasible solution, the dual cannot be unbounded.

Testing if a Basis is Optimal

- What are the costs in the basic feasible solution?

$$cx = c_B A_B^{-1} b + (c - c_B A_B^{-1} A)x$$

$$cx = \Pi b + (c - \Pi A)x$$

- The basis is optimal if these costs are non-negative
- So the simplex multiplier are a feasible solution to the dual

Duality

- Theorem: If the primal has an optimal solution, the dual has an optimal solution with the same cost

Consider the optimal solution x^* .

It has an associated basis B

$$x_B^* = A_B^{-1}b.$$

The dual has a feasible solution

$$y^* = c_B A_B^{-1}$$

by the optimality of the primal. Hence,

$$y^*b = c_B A_B^{-1}b = c_B x^*.$$

General Form of the Dual

min $c x$

subject to

$$a_i x = b_i \quad (i \in E)$$

$$a_i x \geq b_i \quad (i \in I)$$

$$x_j \geq 0 \quad (j \in P)$$

$$x_j \in \mathcal{R} \quad (j \in O)$$

Primal

max $y b$

subject to

$$y_i \in \mathcal{R} \quad (i \in E)$$

$$y_i \geq 0 \quad (i \in I)$$

$$y A_j \leq c_j \quad (j \in P)$$

$$y A_j = c_j \quad (j \in O)$$

Dual

Properties of Duality

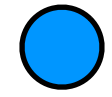
- The dual of the dual is the primal

	Finite Primal	Unbounded Primal	Infeasible Primal
Finite Dual	Yes	?	?
Unbounded Dual	?	?	?
Infeasible Dual	?	?	?

Primal and Dual

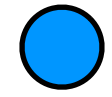
Primal and Dual

Primal

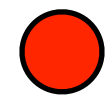


Primal and Dual

Primal

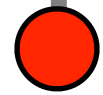
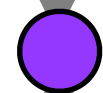
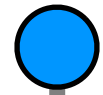


Dual



Primal and Dual

Primal



Dual



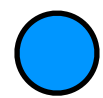
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- The dual of the dual is the primal

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Primal and Dual

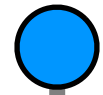
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Dual

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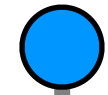
Primal



Dual

Primal and Dual

Primal



Let x and Π be feasible solutions
to the primal and dual respectively.
We have that $cx \geq \Pi Ax \geq \Pi b$.

Dual



Properties of Duality

► The dual of the dual is the primal

	Finite Primal	Unbounded Primal	Infeasible Primal
Finite Dual	Yes	?	?
Unbounded Dual	?	?	?
Infeasible Dual	?	?	?

Primal / Dual Relationships

$$\begin{array}{ll}\min & x_1 \\ \text{subject to} & \\ & x_1 + x_2 \geq 1 \\ & -x_1 - x_2 \geq 1\end{array}$$

infeasible primal

$$\begin{array}{ll}\max & y_1 + y_2 \\ \text{subject to} & \\ & y_1 - y_2 = 1 \\ & y_1 - y_2 = 0 \\ & y_i \geq 0\end{array}$$

infeasible dual

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unbounded dual

Certificate of Optimality

- ▶ NP-Complete Problems
 - certificate of feasibility

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- ▶ Can you provide
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- ▶ NP-Complete Problems
 - certificate of feasibility
- ▶ Can you provide
 - a certificate of optimality?
- ▶ Consider now a linear program.
 - can you convince me that you have found an optimal solution?

Certificate of Optimality

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⋮

dual

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Certificate of Optimality

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► Give me a x^* that satisfies $A x^* \geq b$

Certificate of Optimality

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- ▶ Give me a y^* that satisfies $y^* A \leq c$

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⋮

dual

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- ▶ Give me a x^* that satisfies $A x^* \geq b$
- ▶ Give me a y^* that satisfies $y^* A \leq c$
- ▶ Show me that $c x^* = y^* b$.

Until Next Time