Discrete Optimization

Linear Programming: Part IV

Goals of the Lecture

- Linear programming
  - introduce matrix notations
  - introduce the tableau
  - restate some of the results

Matrix Notations

$$\begin{array}{ccccccccccccc}
3x_1 & + & 2x_2 & + & x_3 & & & & & & & = & 1 \\
2x_1 & & & & & + & x_4 & & & + & x_6 & = & 2 \\
x_1 & & & & & & & + & x_5 & + & x_6 & = & 3
\end{array}$$

Matrix Notations

$$
\begin{array}{rcrcrcrcrcrcl}
3x_1 & + & 2x_2 & + & x_3 & & & & & & & = & 1 \\
2x_1 & & & & & + & x_4 & & & + & x_6 & = & 2 \\
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\end{array}
$$

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Basis = {3,4,5}

Matrix Notations

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\begin{array}{ccccccccccccc}
3x_1 & + & 2x_2 & + & x_3 & & & & & & & = & 1 \\
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x_1 & & & & & & & + & x_5 & + & x_6 & = & 3
\end{array}
$$

Basis = {3,4,5}

$$
\begin{array}{ccccccccc}
x_3 & = & 1 & - & 3x_1 & - & 2x_2 & & \\
x_4 & = & 2 & - & 2x_1 & & & - & x_6 \\
x_5 & = & 3 & & & & & - & x_6
\end{array}
$$

Matrix Notations

$$\begin{array}{ccccccccccccc} 3x_1 & + & 2x_2 & + & x_3 & & & & & & & = & 1 \\ 2x_1 & & & & & + & x_4 & & & + & x_6 & = & 2 \\ x_1 & & & & & & & + & x_5 & + & x_6 & = & 3 \end{array}$$

Basic variables = $\{x_3, x_4, x_5\}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} + \begin{pmatrix} 3 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Matrix Notations

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\begin{array}{ccccccccccccc}
3x_1 & + & 2x_2 & + & x_3 & & & & & & & = & 1 \\
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x_1 & & & & & & & + & x_5 & + & x_6 & = & 3
\end{array}
$$

Basic variables = $\{x_3, x_4, x_5\}$

$$
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix}
+
\begin{pmatrix} 3 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ x_6 \end{pmatrix}
=
\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}
$$

A_B x_B A_N x_N b

How to Obtain a Basic Feasible Solution?

- Consider $Ax = b$
- Choose m linearly independent columns A_B

$$A_B x_B + A_N x_N = b$$
$$A_B x_B = b - A_N x_N$$
$$x_B = A_B^{-1} b - A_B^{-1} A_N x_N$$
$$x_B = b' - A'_N x_N$$

How to Obtain a Basic Feasible Solution?

- Consider $Ax = b$
- Choose m linearly independent columns A_B

$$
\begin{aligned}
&A_B x_B + A_N x_N = b \\
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\end{aligned}
$$

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- Feasible if $b' \geq 0$

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$$A_B x_B + A_N x_N = b$$
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$$x_B = b' - A'_N x_N$$

- Feasible if $b' \geq 0$
- The matrix A_B is called a *basis.*

How to Obtain a Basic Feasible Solution?

$$
\begin{array}{ll}
\min & cx \\
\text{subject to} & \\
 & Ax = b
\end{array}
$$

How to Obtain a Basic Feasible Solution?

- Linear programming

$$\begin{array}{ll} \min & cx \\ \text{subject to} & \\ & Ax = b \end{array}$$

How to Obtain a Basic Feasible Solution?

- Linear programming

$$\begin{aligned} \min \quad & cx \\ \text{subject to} \quad & \\ & Ax = b \end{aligned}$$

- Basic feasible solution: Basis B

$$x_B = A_B^{-1} b - A_B^{-1} A_N x_N$$

How to Obtain a Basic Feasible Solution?

- Linear programming

$$\begin{array}{ll} \min & cx \\ \text{subject to} & \\ & Ax = b \end{array}$$

- Basic feasible solution: Basis B

$$x_B = A_B^{-1} b - A_B^{-1} A_N x_N$$

- What is the cost for basis B?

$$cx \quad = \quad c_B x_B + c_N x_N$$

How to Obtain a Basic Feasible Solution?

$$
\begin{aligned}
cx &= c_B x_B + c_N x_N \\
&= c_B(A_B^{-1} b - A_B^{-1} A_N x_N) + c_N x_N \\
&= c_B A_B^{-1} b + (c_N - c_B A_B^{-1} A_N) x_N \\
&= c_B A_B^{-1} b + (c_N - c_B A_B^{-1} A_N) x_N \\
&\quad + (c_B - c_B A_B^{-1} A_B) x_B \\
&= c_B A_B^{-1} b + (c - c_B A_B^{-1} A) x
\end{aligned}
$$

How to Obtain a Basic Feasible Solution?

- What is the cost for basis B?

$$
\begin{aligned}
cx &= c_B x_B + c_N x_N \\
&= c_B(A_B^{-1}b - A_B^{-1}A_N x_N) + c_N x_N \\
&= c_B A_B^{-1} b + (c_N - c_B A_B^{-1} A_N) x_N \\
&= c_B A_B^{-1} b + (c_N - c_B A_B^{-1} A_N) x_N \\
&\quad +(c_B - c_B A_B^{-1} A_B) x_B \\
&= c_B A_B^{-1} b + (c - c_B A_B^{-1} A) x
\end{aligned}
$$

How to Obtain a Basic Feasible Solution?

- What is the cost for basis B?

$$\begin{aligned} cx &= c_B x_B + c_N x_N \\ &= c_B(A_B^{-1}b - A_B^{-1}A_N x_N) + c_N x_N \\ &= c_B A_B^{-1} b + (c_N - c_B A_B^{-1} A_N) x_N \\ &= c_B A_B^{-1} b + (c_N - c_B A_B^{-1} A_N) x_N \\ &\quad + (c_B - c_B A_B^{-1} A_B) x_B \\ &= c_B A_B^{-1} b + (c - c_B A_B^{-1} A) x \end{aligned}$$

- Define $\Pi = c_B A_B^{-1}$

How to Obtain a Basic Feasible Solution?

- What is the cost for basis B?

$$
\begin{aligned}
cx &= c_B x_B + c_N x_N \\
&= c_B(A_B^{-1}b - A_B^{-1}A_N x_N) + c_N x_N \\
&= c_B A_B^{-1} b + (c_N - c_B A_B^{-1} A_N) x_N \\
&= c_B A_B^{-1} b + (c_N - c_B A_B^{-1} A_N) x_N \\
&\quad + (c_B - c_B A_B^{-1} A_B) x_B \\
&= c_B A_B^{-1} b + (c - c_B A_B^{-1} A) x
\end{aligned}
$$

- Define $\Pi = c_B A_B^{-1}$

$$cx = \Pi b + (c - \Pi A)x$$

Testing if a Basis is Optimal

$$cx = c_B A_B^{-1} b + (c - c_B A_B^{-1} A)x$$

$$cx = \Pi b + (c - \Pi A)x$$

Testing if a Basis is Optimal

- What are the costs in the basic feasible solution?

$$cx \quad = \quad c_B A_B^{-1} b + (c - c_B A_B^{-1} A)x$$

$$cx = \Pi b + (c - \Pi A)x$$

Testing if a Basis is Optimal

- What are the costs in the basic feasible solution?

$$cx \quad = \quad c_B A_B^{-1} b + \boxed{(c - c_B A_B^{-1} A)} x$$

$$cx = \Pi b + \boxed{(c - \Pi A)} x$$

Testing if a Basis is Optimal

- What are the costs in the basic feasible solution?

$$cx = c_B A_B^{-1} b + \boxed{(c - c_B A_B^{-1} A)}x$$

$$cx = \Pi b + \boxed{(c - \Pi A)}x$$

- The basis is optimal if these costs are non-negative

Testing if a Basis is Optimal

Let $c^* = c - \Pi A$ and $c_0^* = \Pi b$.
Since the c^* are nonnegative,

$$c \geq \Pi A.$$

Consider any feasible solution y. We have

$$Ay = b.$$

Hence,

$$cy \geq \Pi A y = \Pi b = c_0^*.$$

The Tableau

- Linear programming is often presented with a tableau
  - easier for pivoting

$$
\begin{array}{lrcrcrcrcrcl}
\text{min} & x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & = & z \\
\text{subject to} & & & & & & & & & & & \\
 & 3x_1 & + & 2x_2 & + & x_3 & & & & & = & 1 \\
 & 5x_1 & + & x_2 & + & x_3 & + & x_4 & & & = & 3 \\
 & 2x_1 & + & 5x_2 & + & x_3 & & & + & x_5 & = & 4
\end{array}
$$

The Tableau

$$
\begin{array}{lccccccccccc}
\text{min} & x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & = & z \\
\text{subject to} & & & & & & & & & & & \\
 & 3x_1 & + & 2x_2 & + & x_3 & & & & & = & 1 \\
 & 5x_1 & + & x_2 & + & x_3 & + & x_4 & & & = & 3 \\
 & 2x_1 & + & 5x_2 & + & x_3 & & & + & x_5 & = & 4
\end{array}
$$

x_1	x_2	x_3	x_4	x_5	$-z$
-3	-3	0	0	0	-6
3	2	1	0	0	1
2	-1	0	1	0	2
-1	3	0	0	1	3

The Tableau

$$
\begin{array}{lrcrcrcrcrcl}
\text{min} & x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & = & z \\
\text{subject to} \\
& 3x_1 & + & 2x_2 & + & x_3 & & & & & = & 1 \\
& 5x_1 & + & x_2 & + & x_3 & + & x_4 & & & = & 3 \\
& 2x_1 & + & 5x_2 & + & x_3 & & & + & x_5 & = & 4
\end{array}
$$

x_1	x_2	x_3	x_4	x_5	-z
-3	-3	0	0	0	-6
3	2	1	0	0	1
2	-1	0	1	0	2
-1	3	0	0	1	3

-z

The Tableau

$$\begin{array}{lrcrcrcrcrcl} \text{min} & x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & = & z \\ \text{subject to} & & & & & & & & & & & \\ & 3x_1 & + & 2x_2 & + & x_3 & & & & & = & 1 \\ & 5x_1 & + & x_2 & + & x_3 & + & x_4 & & & = & 3 \\ & 2x_1 & + & 5x_2 & + & x_3 & & & + & x_5 & = & 4 \end{array}$$

x_1	x_2	x_3	x_4	x_5	-z
-3	-3	0	0	0	-6
3	2	1	0	0	1
2	-1	0	1	0	2
-1	3	0	0	1	3

-z

the basis

The Tableau

$$
\begin{array}{llllllllllll}
\text{min} & x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & = & z \\
\text{subject to} & & & & & & & & & & & \\
 & 3x_1 & + & 2x_2 & + & x_3 & & & & & = & 1 \\
 & 5x_1 & + & x_2 & + & x_3 & + & x_4 & & & = & 3 \\
 & 2x_1 & + & 5x_2 & + & x_3 & & & + & x_5 & = & 4
\end{array}
$$

x_1	x_2	x_3	x_4	x_5	-z
-3	-3	0	0	0	-6
3	2	1	0	0	1
2	-1	0	1	0	2
-1	3	0	0	1	3

-z

the basis

b

The Tableau

x_1	x_2	x_3	x_4	x_5	$-z$
-3	-3	0	0	0	-6
3	2	1	0	0	1
2	-1	0	1	0	2
-1	3	0	0	1	3

The Tableau

entering variable

x_1	x_2	x_3	x_4	x_5	-z
-3	-3	0	0	0	-6
3	2	1	0	0	1
2	-1	0	1	0	2
-1	3	0	0	1	3

The Tableau

entering variable

x_1	x_2	x_3	x_4	x_5	$-z$
-3	-3	0	0	0	-6
3	2	1	0	0	1
2	-1	0	1	0	2
-1	3	0	0	1	3

The Tableau

entering variable

x_1	x_2	x_3	x_4	x_5	$-z$
-3	-3	0	0	0	-6
3	2	1	0	0	1
2	-1	0	1	0	2
-1	3	0	0	1	3

x_1	x_2	x_3	x_4	x_5	$-z$
3/2	0	3/2	0	0	-9/2
3/2	1	1/2	0	0	1/2
7/2	0	1/2	1	0	5/2
-11/2	0	-3/2	0	1	3/2

Until Next Time