

Discrete Optimization

Linear Programming: Part II

Goals of the Lecture

- ▶ Linear programming
 - algebraic view
 - links with geometry

Geometry of Linear Programming

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- ▶ How to solve a linear program “geometrically”?
 - enumerate all the vertices
 - select the one with the smallest objective value

Geometry of Linear Programming

- ▶ How to solve a linear program “geometrically”?
 - enumerate all the vertices
 - select the one with the smallest objective value
- ▶ The simplex algorithm
 - a more intelligent way of exploring the vertices
 - connection between the algebraic and geometrical view

The Simplex Algorithm

- ▶ Invented by G. Dantzig

The Simplex Algorithm

- ▶ Invented by G. Dantzig
- ▶ Very interesting algorithm
 - works incredibly well in practice
 - exponential worst-case
 - a real theoretical enigma

Outline of the Simplex Algorithm

Goal: You want to be on top of the world

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2. The top of a mountain is a Beautiful Fantastic Spot (BFS).

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3. You can move from one BFS to a neighboring BFS

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2. The top of a mountain is a Beautiful Fantastic Spot (BFS).
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4. You can see that you are on top of the world

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1. The top of the world is the top of a mountain
2. The top of a mountain is a Beautiful Fantastic Spot (BFS).
3. You can move from one BFS to a neighboring BFS
4. You can see that you are on top of the world
5. From any BFS, you can move to a higher BFS

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4. You can detect whether a BFS is optimal

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3. You can move from one BFS to a neighboring BFS
4. You can detect whether a BFS is optimal
5. From any BFS, you can move to a BFS with a better cost

Linear Programs

$$\min c_1 x_1 + \dots + c_n x_n$$

subject to

$$a_{11}x_1 + \dots + a_{1n}x_n \leq b_1$$

...

$$a_{m1}x_1 + \dots + a_{mn}x_n \leq b_m$$

$$x_i \geq 0 \quad (1 \leq i \leq n)$$

Algebraic View: BFS

Goal: How to find solutions to linear systems

$$a_{11}x_1 + \dots + a_{1n}x_n = b_1$$

...

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Algebraic View: BFS

Basic Solution

$$\begin{aligned}x_1 &= b_1 + \sum_{i=m+1}^n a_{1i} x_i \\ \dots \\ x_m &= b_m + \sum_{i=m+1}^n a_{mi} x_i\end{aligned}$$

Algebraic View: BFS

Basic Solution

$$\begin{array}{l} x_1 \\ \dots \\ x_m \end{array} = \begin{array}{l} b_1 + \sum_{i=m+1}^n a_{1i} x_i \\ \dots \\ b_m + \sum_{i=m+1}^n a_{mi} x_i \end{array}$$

Basic Variables

Algebraic View: BFS

Basic Solution

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Basic Variables

Non Basic Variables

Algebraic View: BFS

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Assign to the b's

Assign to zero

Algebraic View: BFS

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$$\{x_i = b_i \mid 1 \leq i \leq m\} \cup \{x_i = 0 \mid m+1 \leq i \leq n\}$$

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Basic Feasible Solution

$$\begin{aligned}x_1 &= b_1 + \sum_{i=m+1}^n a_{1i} x_i \\&\dots \\x_m &= b_m + \sum_{i=m+1}^n a_{mi} x_i\end{aligned}$$

Feasible if $\forall i \in 1..m : b_i \geq 0$

Finding BFS

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- ▶ If the b 's are all nonnegative
 - we have a basic feasible solution

Sir, we do not have equations.

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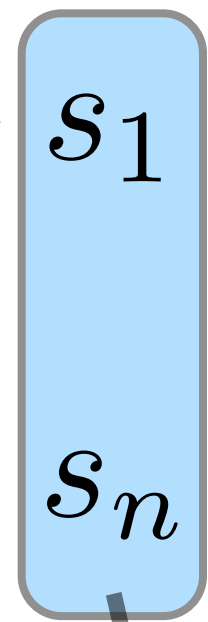
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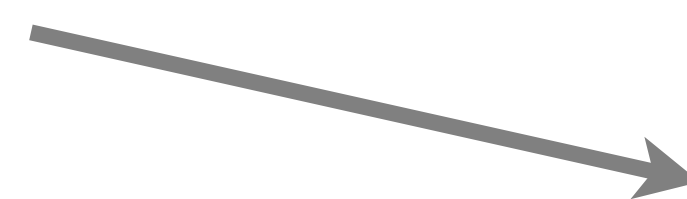
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Slack Variables

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 - by adding slack variables

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– generate all basic feasible solutions

- select m basic variables and perform Gaussian elimination
- test whether it is feasible

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 - test whether it is feasible
- select the BFS with the best cost

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► How many basic solutions?

$$\frac{n!}{m!(n-m)!}$$

Until Next Time