Discrete Optimization

Linear Programming: Part II

Goals of the Lecture

- Linear programming
  - algebraic view
  - links with geometry

Geometry of Linear Programming

Geometry of Linear Programming

- How to solve a linear program "geometrically"?
  - enumerate all the vertices
  - select the one with the smallest objective value

Geometry of Linear Programming

- How to solve a linear program “geometrically”?
  - enumerate all the vertices
  - select the one with the smallest objective value
- The simplex algorithm
  - a more intelligent way of exploring the vertices
  - connection between the algebraic and geometrical view

The Simplex Algorithm

- Invented by G. Dantzig

The Simplex Algorithm

- Invented by G. Dantzig

- Very interesting algorithm
  - works incredibly well in practice
  - exponential worst-case
  - a real theoretical enigma

Outline of the Simplex Algorithm

Goal: You want to be on top of the world

Outline of the Simplex Algorithm

Goal: You want to be on top of the world

1. The top of the world is the top of a mountain

Outline of the Simplex Algorithm

Goal: You want to be on top of the world

1. The top of the world is the top of a mountain
2. The top of a mountain is a Beautiful Fantastic Spot (BFS).

Outline of the Simplex Algorithm

Goal: You want to be on top of the world

1. The top of the world is the top of a mountain
2. The top of a mountain is a Beautiful Fantastic Spot (BFS).
3. You can move from one BFS to a neighboring BFS

Outline of the Simplex Algorithm

Goal: You want to be on top of the world

1. The top of the world is the top of a mountain
2. The top of a mountain is a Beautiful Fantastic Spot (BFS).
3. You can move from one BFS to a neighboring BFS
4. You can see that you are on top of the world

Outline of the Simplex Algorithm

Goal: You want to be on top of the world

1. The top of the world is the top of a mountain
2. The top of a mountain is a Beautiful Fantastic Spot (BFS).
3. You can move from one BFS to a neighboring BFS
4. You can see that you are on top of the world
5. From any BFS, you can move to a higher BFS

Outline of the Simplex Algorithm

Goal: You want to solve a linear program

Outline of the Simplex Algorithm

Goal: You want to solve a linear program

1. An optimal solution is located at a vertex.

Outline of the Simplex Algorithm

Goal: You want to solve a linear program

1. An optimal solution is located at a vertex.
2. A vertex is a Basic Feasible Solution (BFS).

Outline of the Simplex Algorithm

Goal: You want to solve a linear program

1. An optimal solution is located at a vertex.
2. A vertex is a Basic Feasible Solution (BFS).
3. You can move from one BFS to a neighboring BFS

Outline of the Simplex Algorithm

Goal: You want to solve a linear program

1. An optimal solution is located at a vertex.
2. A vertex is a Basic Feasible Solution (BFS).
3. You can move from one BFS to a neighboring BFS
4. You can detect whether a BFS is optimal

Outline of the Simplex Algorithm

Goal: You want to solve a linear program

1. An optimal solution is located at a vertex.
2. A vertex is a Basic Feasible Solution (BFS).
3. You can move from one BFS to a neighboring BFS
4. You can detect whether a BFS is optimal
5. From any BFS, you can move to a BFS with a better cost

Linear Programs

$$\min c_1 x_1 + \ldots + c_n x_n$$

subject to

$$a_{11} x_1 + \ldots + a_{1n} x_n \leq b_1$$

$$\ldots$$

$$a_{m1} x_1 + \ldots + a_{mn} x_n \leq b_m$$

$$x_i \geq 0 \quad (1 \leq i \leq n)$$

Algebraic View: BFS

Goal: How to find solutions to linear systems

$$\mathrm{a}_{11}x_1 + \ldots + a_{1n}x_n = b_1$$

$$\ldots$$

$$a_{m1}x_1 + \ldots + a_{mn}x_n = b_m$$

$$x_i \geq 0 \quad (1 \leq i \leq n)$$

Algebraic View: BFS

Basic Solution

$$x_1 = b_1 + \sum_{i=m+1}^{n} a_{1i}\, x_i$$

$$\ldots$$

$$x_m = b_m + \sum_{i=m+1}^{n} a_{mi}\, x_i$$

Algebraic View: BFS

Basic Solution

$$x_1 = b_1 + \sum_{i=m+1}^{n} a_{1i}\, x_i$$

$$\dots$$

$$x_m = b_m + \sum_{i=m+1}^{n} a_{mi}\, x_i$$

Basic Variables

Algebraic View: BFS

Basic Solution

$$x_1 = b_1 + \sum_{i=m+1}^{n} a_{1i}\, x_i$$

$$\dots$$

$$x_m = b_m + \sum_{i=m+1}^{n} a_{mi}\, x_i$$

Basic Variables

Non Basic Variables

Algebraic View: BFS

Basic Solution

$$x_1 = b_1 + \sum_{i=m+1}^{n} a_{1i}\, x_i$$

$$\ldots$$

$$x_m = b_m + \sum_{i=m+1}^{n} a_{mi}\, x_i$$

Assign to the b's

Assign to zero

Algebraic View: BFS

Basic Solution

$$x_1 = b_1 + \sum_{i=m+1}^{n} a_{1i}\, x_i$$

$$\dots$$

$$x_m = b_m + \sum_{i=m+1}^{n} a_{mi}\, x_i$$

$$\{x_i = b_i | 1 \leq i \leq m\} \cup \{x_i = 0 | m+1 \leq i \leq n\}$$

Algebraic View: BFS

Goal: How to find solutions to linear systems

$$a_{11}x_1 + \ldots + a_{1n}x_n = b_1$$

$$\ldots$$

$$a_{m1}x_1 + \ldots + a_{mn}x_n = b_m$$

$$x_i \geq 0 \quad (1 \leq i \leq n)$$

Algebraic View: BFS

Basic Solution

$$x_1 = b_1 + \sum_{i=m+1}^{n} a_{1i}\, x_i$$

$$\dots$$

$$x_m = b_m + \sum_{i=m+1}^{n} a_{mi}\, x_i$$

Basic Variables

Non Basic Variables

Algebraic View: BFS

Basic Feasible Solution

$$x_1 = \boxed{b_1} + \sum_{i=m+1}^{n} a_{1i}\, x_i$$

$$\dots$$

$$x_m = \boxed{b_m} + \sum_{i=m+1}^{n} a_{mi}\, x_i$$

Feasible if $\forall i \in 1..m : b_i \geq 0$

Finding BFS

Finding BFS

- Select m variables
  - these will be the basic variables

Finding BFS

- Select m variables
  - these will be the basic variables
- Re-express them in terms of the non-basic variables only
  - Gaussian elimination

Finding BFS

- Select m variables
  – these will be the basic variables
- Re-express them in terms of the non-basic variables only
  – Gaussian elimination
- If the b's are all nonnegative
  – we have a basic feasible solution

Sir, we do not have equations.

$$\min c_1 x_1 + \ldots + c_n x_n$$

$$\text{subject to}$$

$$a_{11} x_1 + \ldots + a_{1n} x_n \leq b_1$$

$$\ldots$$

$$a_{m1} x_1 + \ldots + a_{mn} x_n \leq b_m$$

$$x_i \geq 0 \quad (1 \leq i \leq n)$$

Sir, we do not have equations.

$$
\begin{aligned}
&\mathrm{a}_{11}\ x_1 + \ldots + a_{1n}\ x_n \leq b_1 \\
&\ldots \\
&a_{m1}x_1 + \ldots + a_{mn}x_n \leq b_m
\end{aligned}
$$

Sir, we do not have equations.

$$a_{11}\ x_1 + \ldots + a_{1n}\ x_n \leq b_1$$

$$\ldots$$

$$a_{m1}x_1 + \ldots + a_{mn}x_n \leq b_m$$

$$\downarrow$$

$$a_{11}\ x_1 + \ldots + a_{1n}\ x_n + s_1 = b_1$$

$$\ldots$$

$$a_{m1}x_1 + \ldots + a_{mn}x_n + s_n = b_m$$

$$s_1, \ldots, s_m \geq 0$$

Sir, we do not have equations.

$$a_{11}\ x_1 + \ldots + a_{1n}\ x_n \leq b_1$$
$$\ldots$$
$$a_{m1}x_1 + \ldots + a_{mn}x_n \leq b_m$$

$$a_{11}\ x_1 + \ldots + a_{1n}\ x_n + \boxed{s_1} = b_1$$
$$\ldots$$
$$a_{m1}x_1 + \ldots + a_{mn}x_n + \boxed{s_n} = b_m$$

$$s_1, \ldots, s_m \geq 0$$

Sir, we do not have equations.

$$a_{11}\ x_1 + \ldots + a_{1n}\ x_n \leq b_1$$
$$\ldots$$
$$a_{m1}x_1 + \ldots + a_{mn}x_n \leq b_m$$

$$a_{11}\ x_1 + \ldots + a_{1n}\ x_n + \boxed{s_1} = b_1$$
$$\ldots$$
$$a_{m1}x_1 + \ldots + a_{mn}x_n + \boxed{s_n} = b_m$$

$$s_1, \ldots, s_m \geq 0$$

Slack Variables

Finding BFS

- Re-express the constraints as equations
  - by adding slack variables

Finding BFS

- Re-express the constraints as equations
  - by adding slack variables
- Select m variables
  - these will be the basic variables

Finding BFS

- Re-express the constraints as equations
  - by adding slack variables
- Select m variables
  - these will be the basic variables
- Re-express them in terms of the non-basic variables only
  - Gaussian elimination

Finding BFS

- Re-express the constraints as equations
  - by adding slack variables
- Select m variables
  - these will be the basic variables
- Re-express them in terms of the non-basic variables only
  - Gaussian elimination
- If the b's are all nonnegative
  - we have a basic feasible solution

And now ladies and gentlemen ...

1. An optimal solution is located at a vertex.
2. A vertex is a Basic Feasible Solution (BFS).

And now ladies and gentlemen ...

1. An optimal solution is located at a vertex.
2. A vertex is a Basic Feasible Solution (BFS).

- Naive algorithm

And now ladies and gentlemen ...

1. An optimal solution is located at a vertex.
2. A vertex is a Basic Feasible Solution (BFS).

- Naive algorithm
  - generate all basic feasible solutions
    - select m basic basic variables and perform Gaussian elimination
    - test whether it is feasible

And now ladies and gentlemen ...

1. An optimal solution is located at a vertex.
2. A vertex is a Basic Feasible Solution (BFS).

- Naive algorithm
  - generate all basic feasible solutions
    - select m basic basic variables and perform Gaussian elimination
    - test whether it is feasible
  - select the BFS with the best cost

And now ladies and gentlemen ...

1. An optimal solution is located at a vertex.
2. A vertex is a Basic Feasible Solution (BFS).

- Naive algorithm
  - generate all basic feasible solutions
    - select m basic basic variables and perform Gaussian elimination
    - test whether it is feasible
  - select the BFS with the best cost
- How many basic solutions?

$$\frac{n!}{m!(n-m)!}$$

Until Next Time