Discrete Optimization

Local Search: Part VII

Goal of the Lecture

- Local search
  - more systematic presentation
    - beyond neighborhood
  - heuristics versus meta-heuristics
  - heuristics

Local Search

- States
  - either solutions or configurations

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- Select one of the legal neighbors
  - S(L(N(s),s),s); selection function

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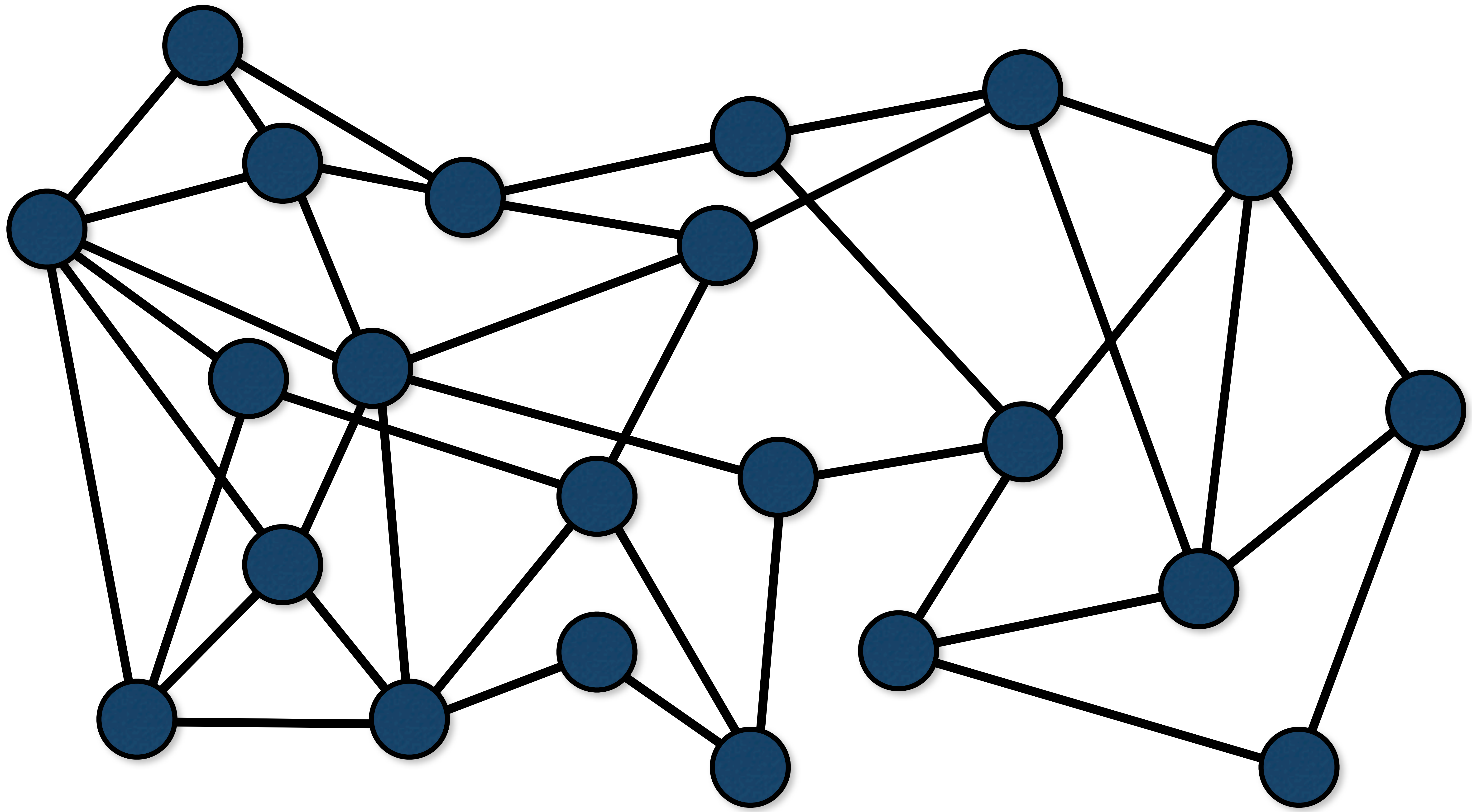
- Some neighbors are legal; others are not
  - L(N(s),s): set of legal neighbors
- Select one of the legal neighbors
  - S(L(N(s),s),s); selection function
- Objective function
  - minimizing f(s)

The Basic Local Search

```
function LocalSearch($f, N, L, S$) {
    $s :=$ generateInitialSolution();
    $s^* := s$;
    for $k := 1$ to $MaxTrials$ do
        if $satisfiable(s) \wedge f(s) < f(s^*)$ then
            $s^* := s$;
        $s := S(L(N(s), s), s)$;
    return $s^*$;
}
```

The Basic Local Search



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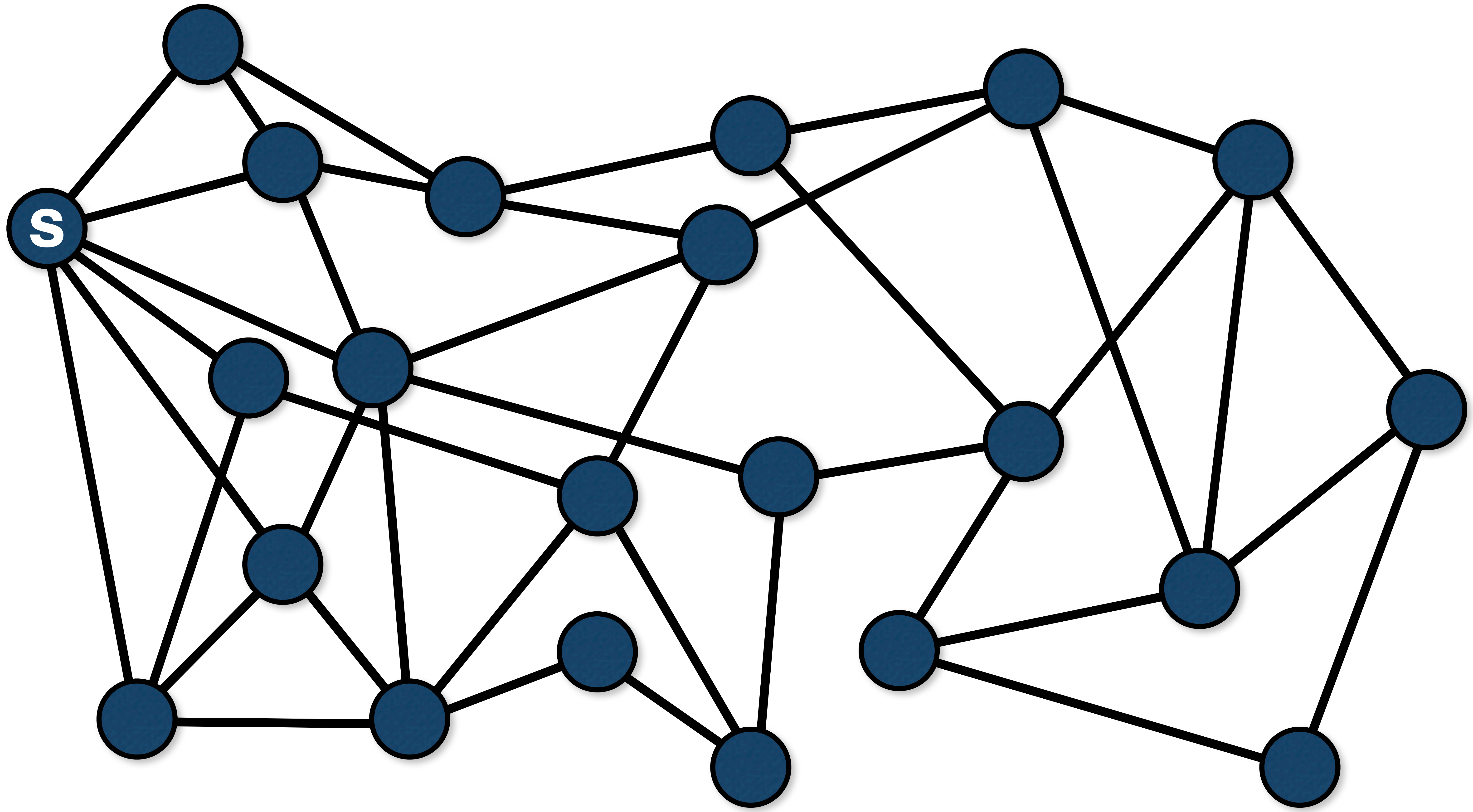

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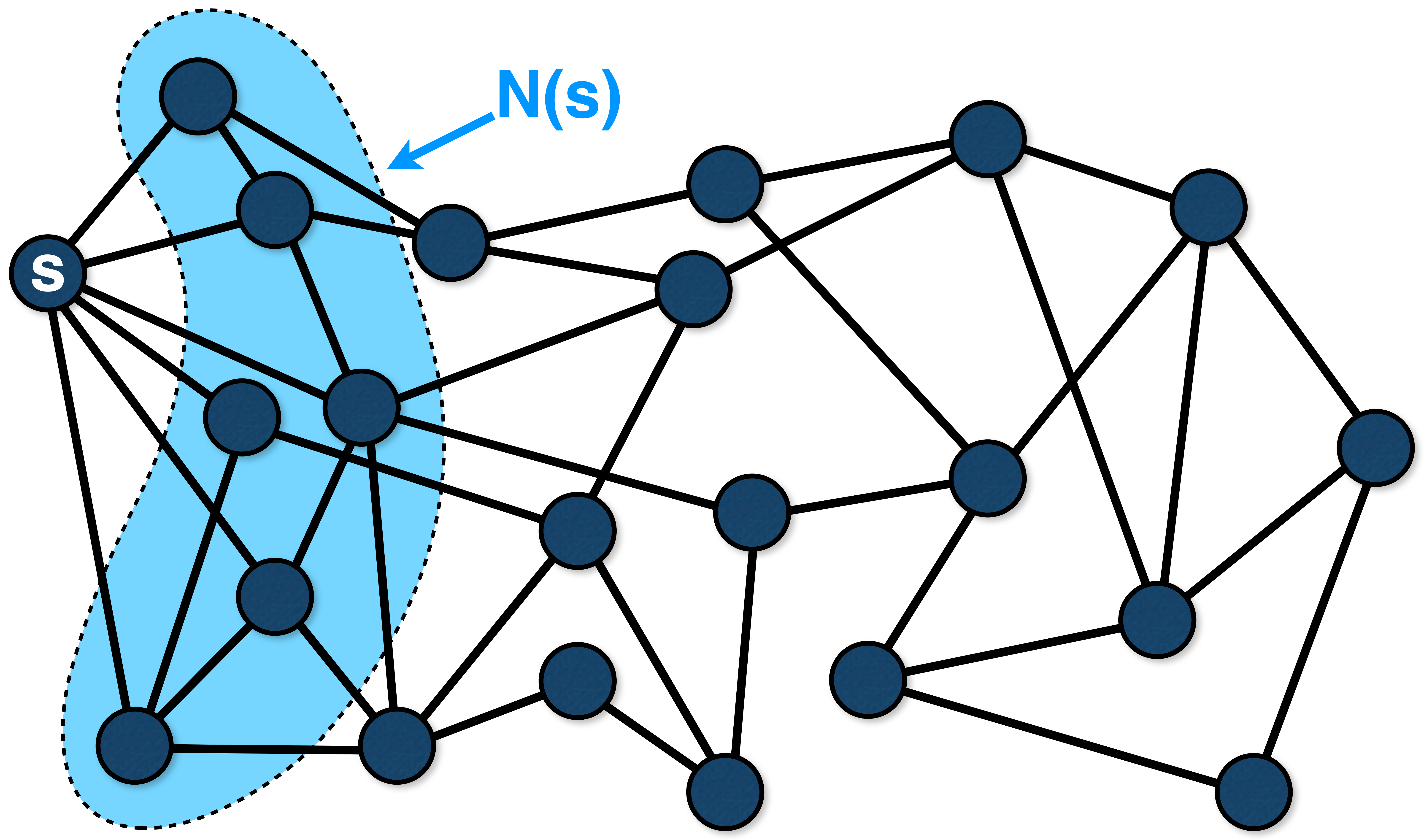

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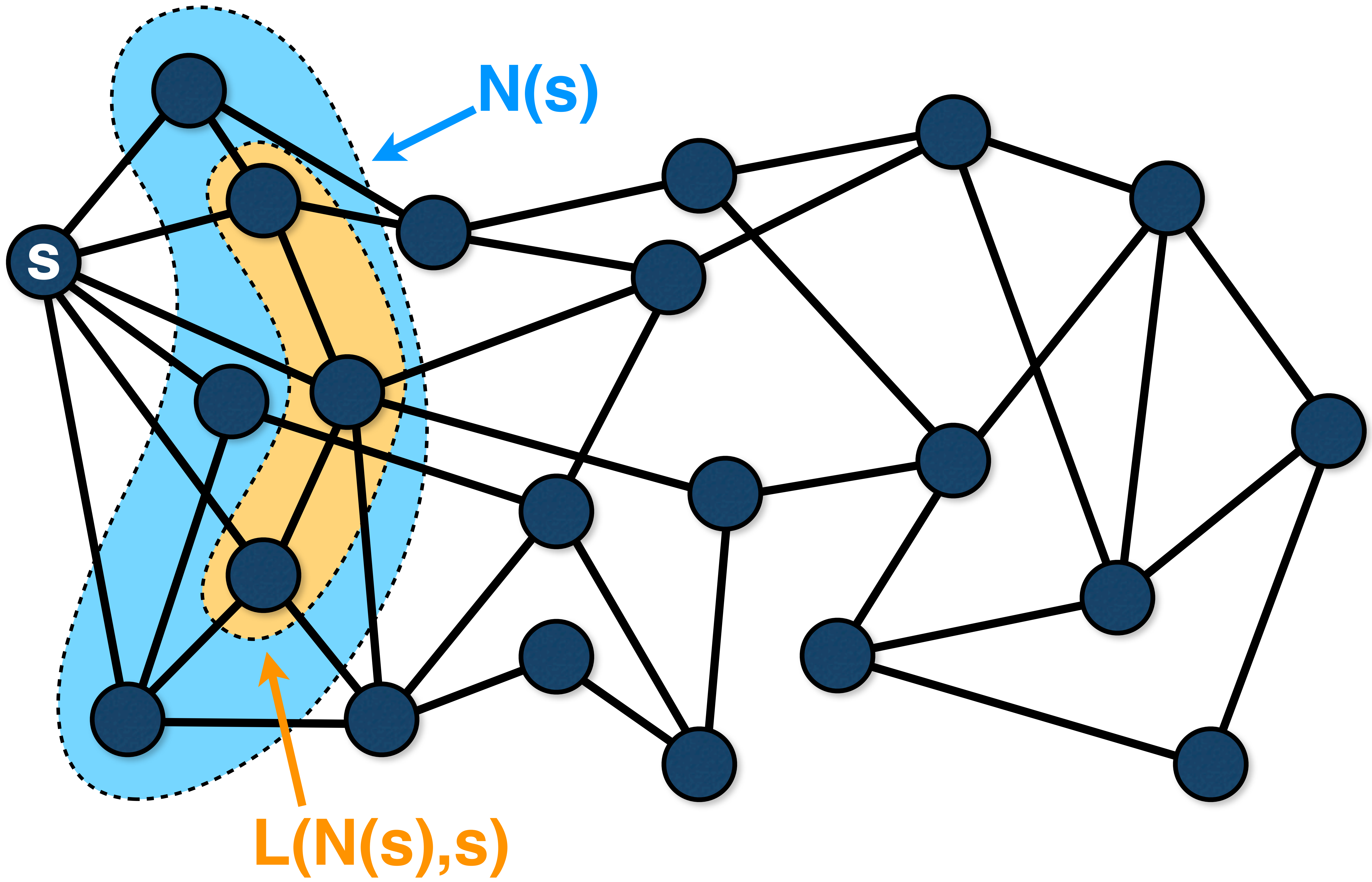

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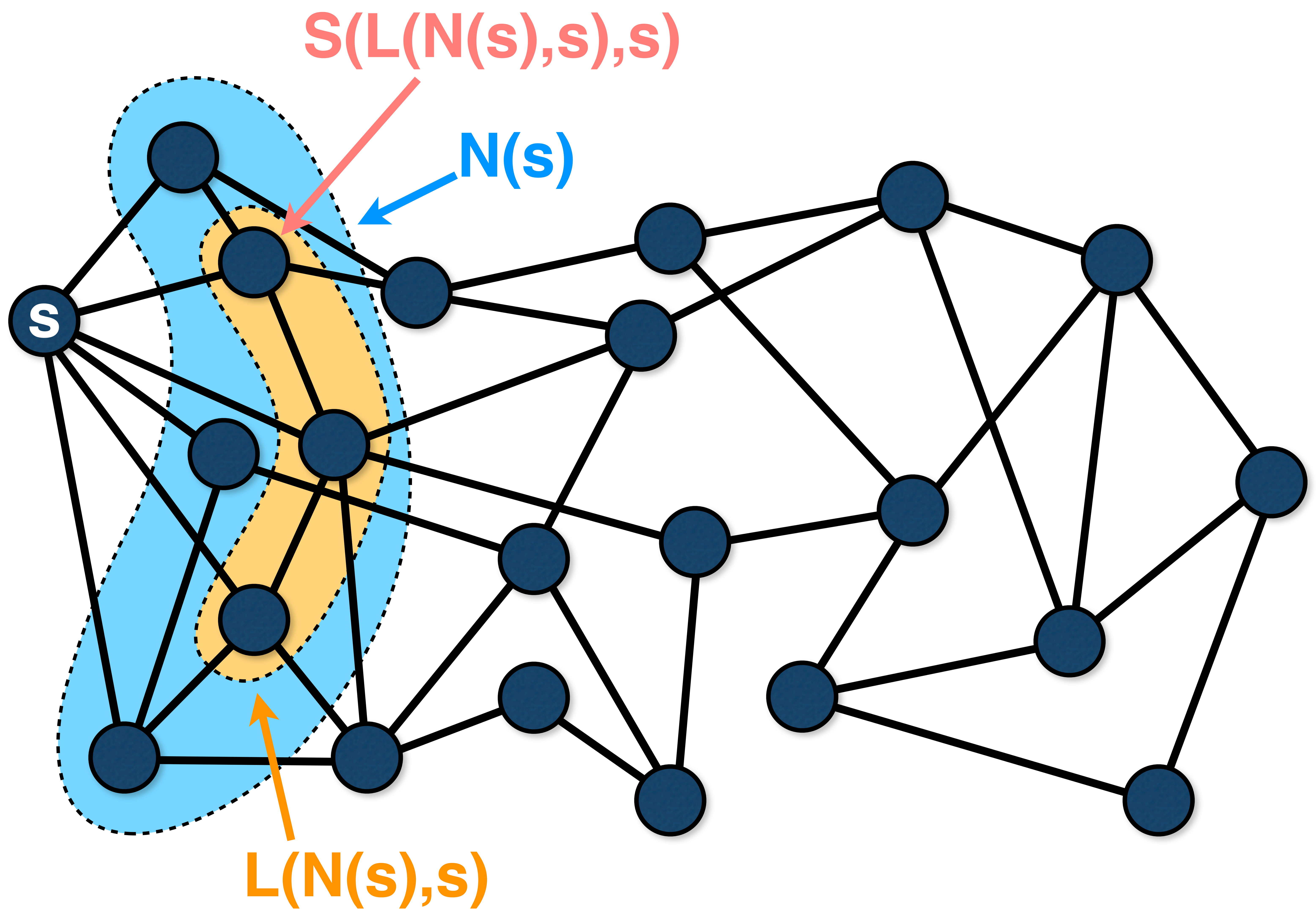

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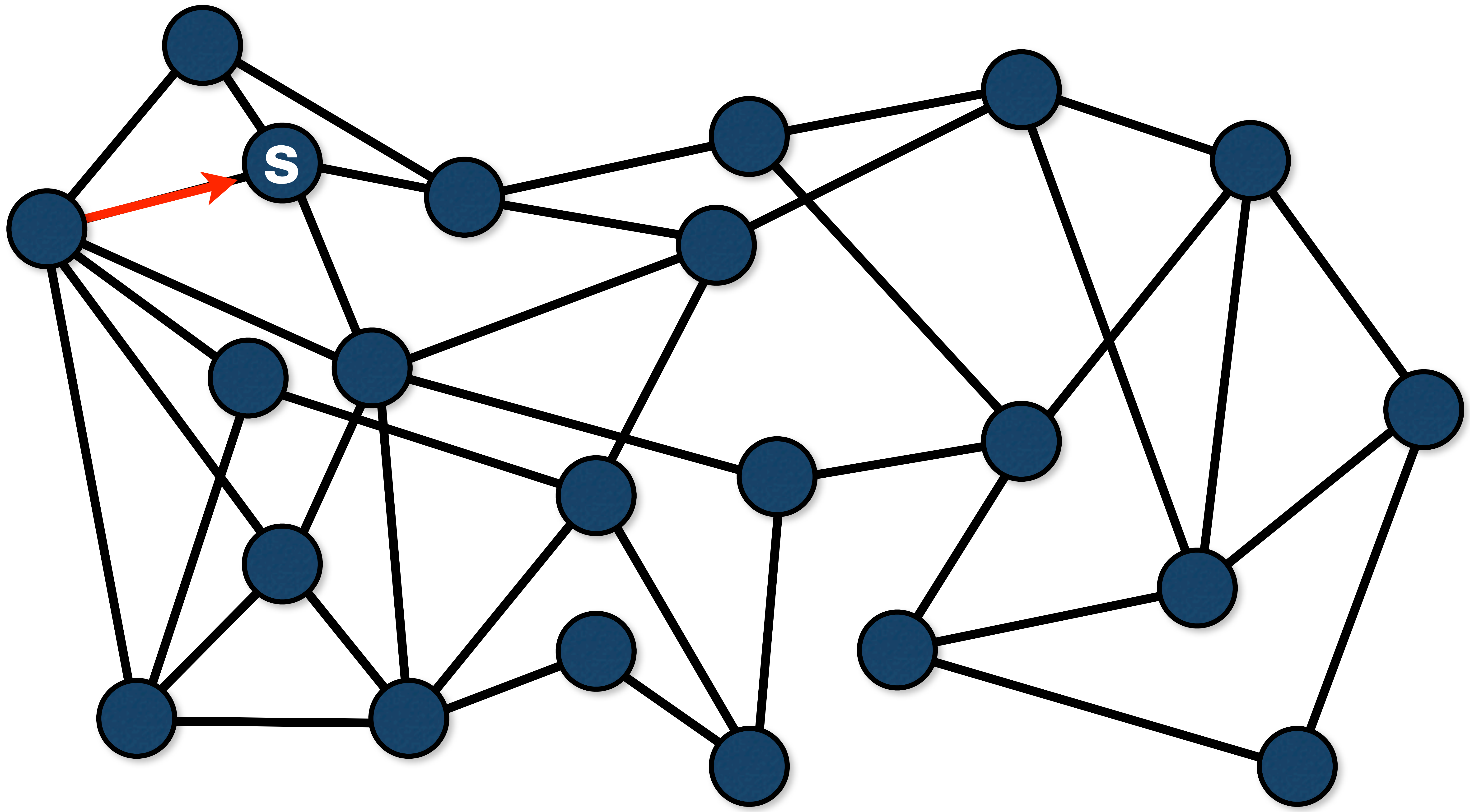

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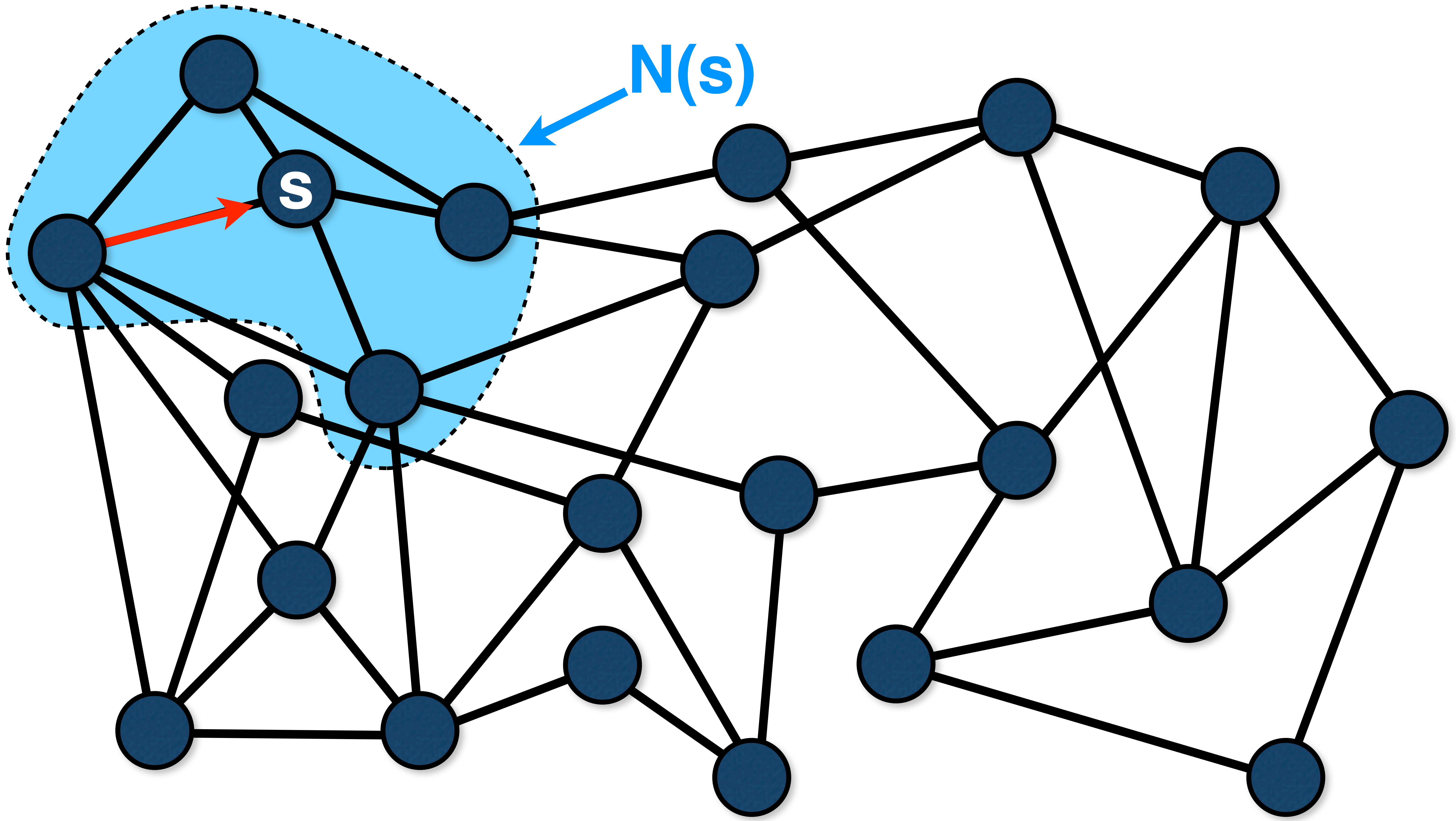

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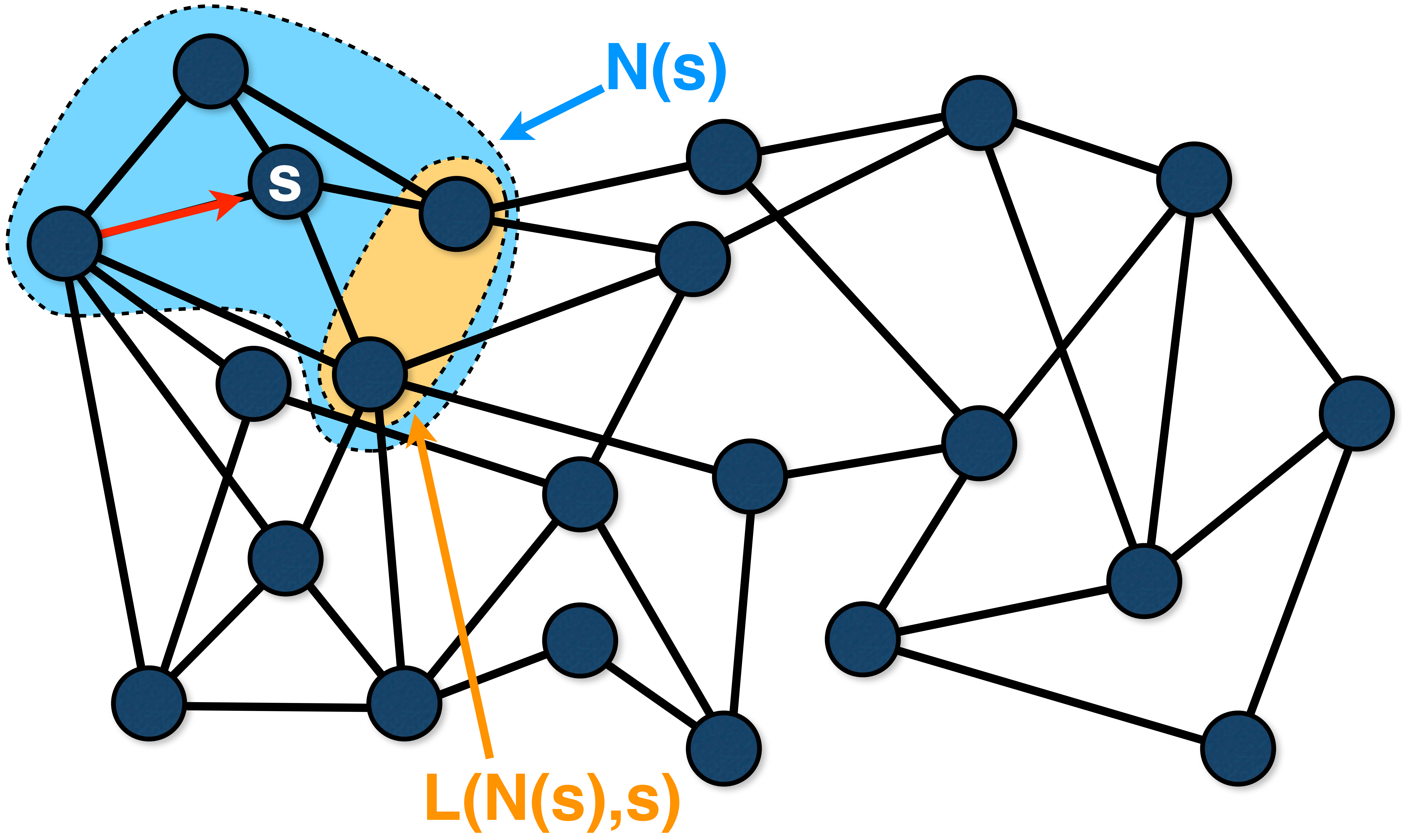

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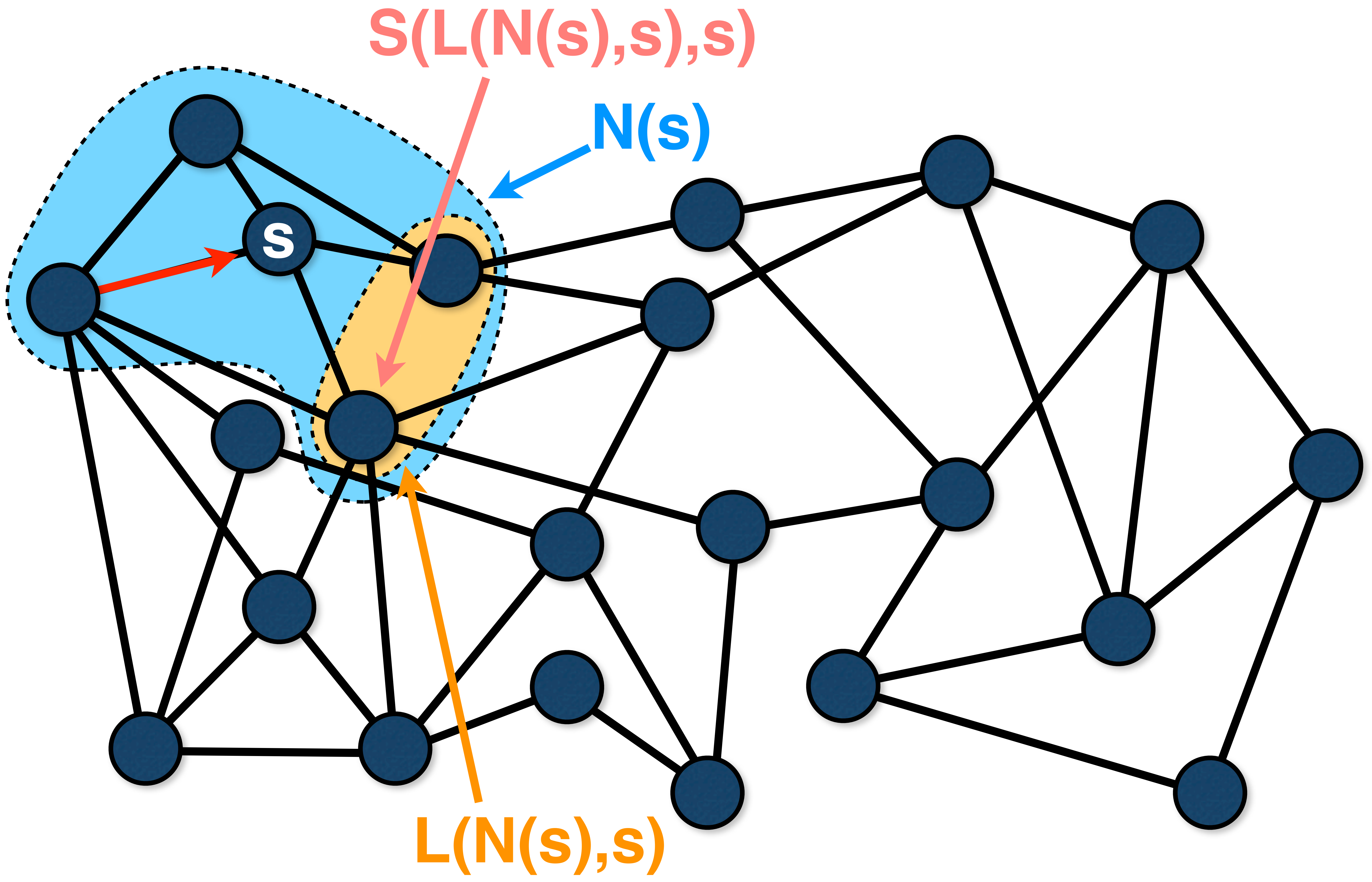

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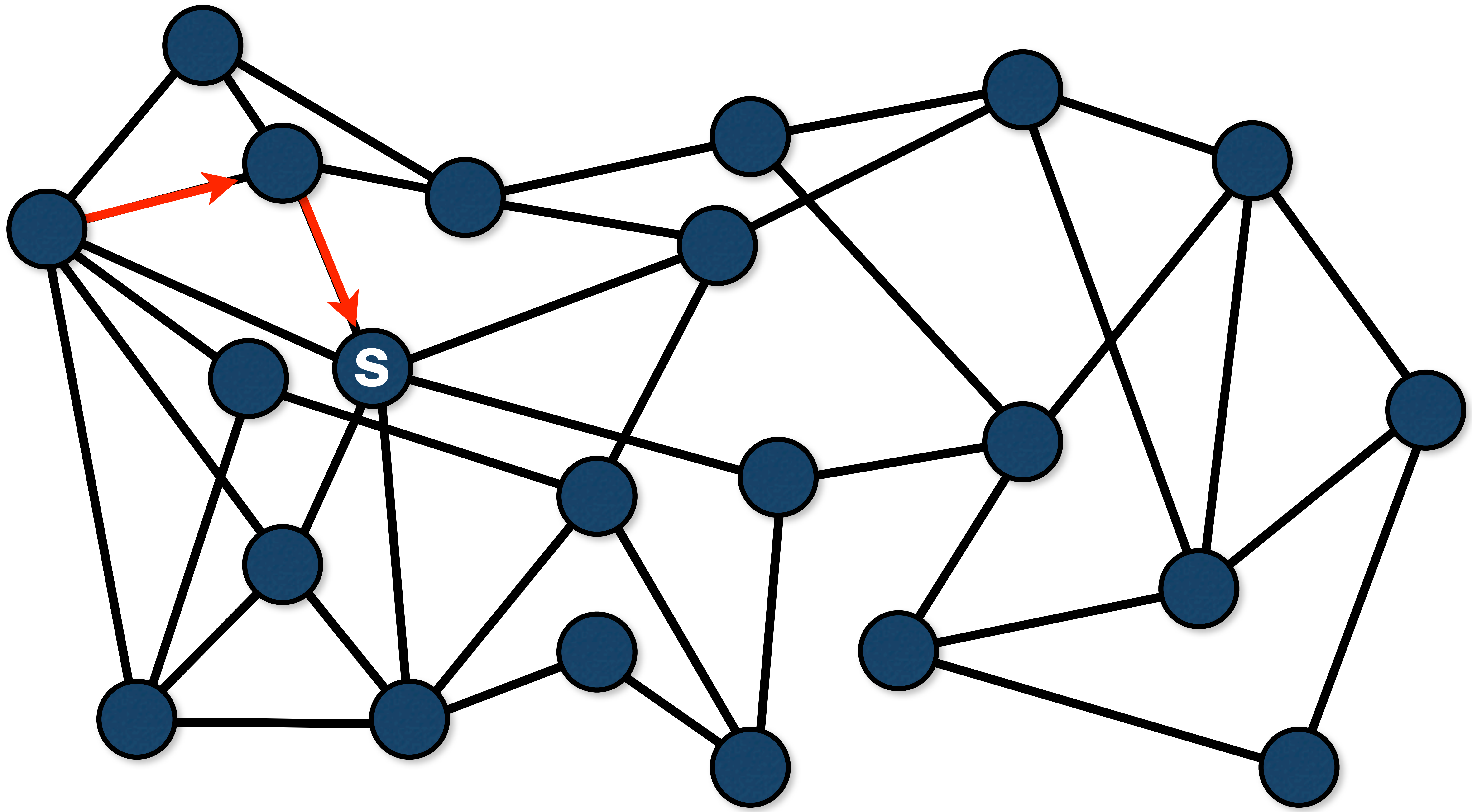

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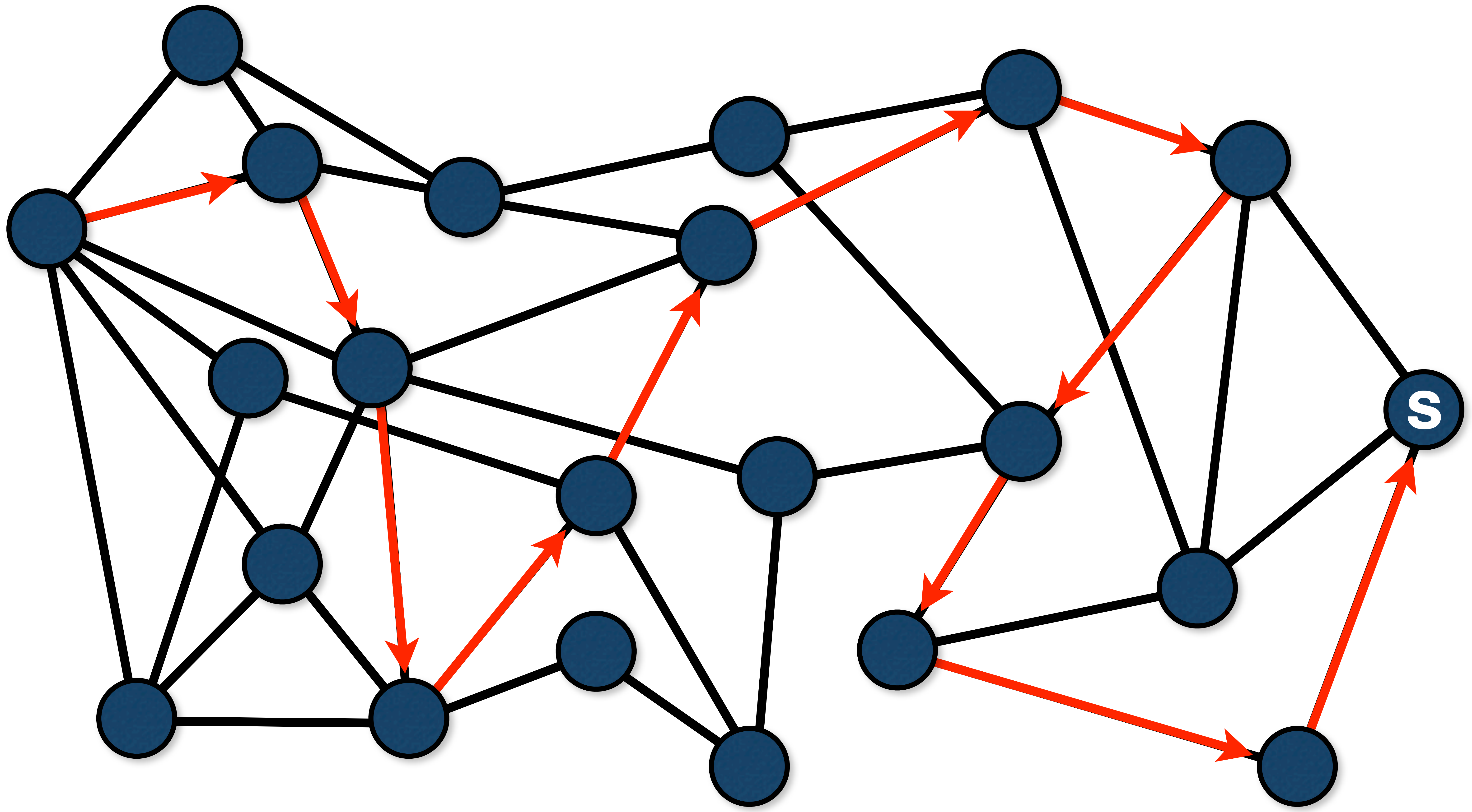

A Example of Local Search

- Legal moves: local improvements
  - $L(N,s) = \{ n \text{ in } N \mid f(n) < f(s) \}$

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- Legal moves: local improvements
  - L(N,s) = { n in N I f(n) < f(s) }

- Selection function: greedy selection
  - S(L,s) = arg-min(n in L) f(n)

Heuristics and Metaheuristics

- Heuristics
  - choose the next neighbor
  - use local information:
    - the state s and its neighborhood
  - drive the search towards a local minimum

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- Heuristics
  - choose the next neighbor
  - use local information:
    - the state s and its neighborhood
  - drive the search towards a local minimum
- Metaheuristics
  - aim at escaping local minima
  - drive the search towards a global minimum
  - typically include some memory or learning

Properties of the Neighbors

- Legal neighbors
  - Conditions on the value of the objective function

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  - $L(N,s) = \{ n \text{ in } N \mid f(n) <= f(s) \}$

Properties of the Neighbors

- Legal neighbors
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- Local improvement
  - L(N,s) = { n in N | f(n) < f(s) }
- No degradation
  - L(N,s) = { n in N | f(n) <= f(s) }
- Potential degradation
  - L(N,s) = N

Selecting a Neighbor

- How to select the neighbor?
  - exploring the whole or part of the neighborhood

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- Best neighbor
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- First neighbor
  - select the first “legal” neighbor in the neighborhood
- Multi-stage selection
  - first select one “part” of neighbor
  - second select the remaining “part” of the neighbor

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- Randomization is often important in local search
  - more on this soon

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```
1. **function** S-Best(N,s)
2.     $N^* := \{\ n \in N \mid f(n) = \min_{s \in N} f(s)\ \}$;
3.     **return** $n \in N^*$ with probability $1/\#N^*$;
```

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```

- Best improvement

```
1. function BESTIMPROVEMENT(s)
2.     return LOCALSEARCH($f$,$N$,L-IMPROVEMENT,S-BEST);
```

First Neighbor

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  - avoid scanning the entire neighborhood

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```
1. **function** S-FIRST(N,s)
2.     **return** $n \in N$ minimizing $lex(n)$;
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First Neighbor

- First neighbor (in some lexicographic order)
  - avoid scanning the entire neighborhood
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```
1. function S-FIRST(N,s)
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```

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```
1. function FIRSTIMPROVEMENT(s)
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```

Multi-Stage Heuristics

- Motivation
  - avoid scanning the entire neighborhood
  - still keep a greedy flavor

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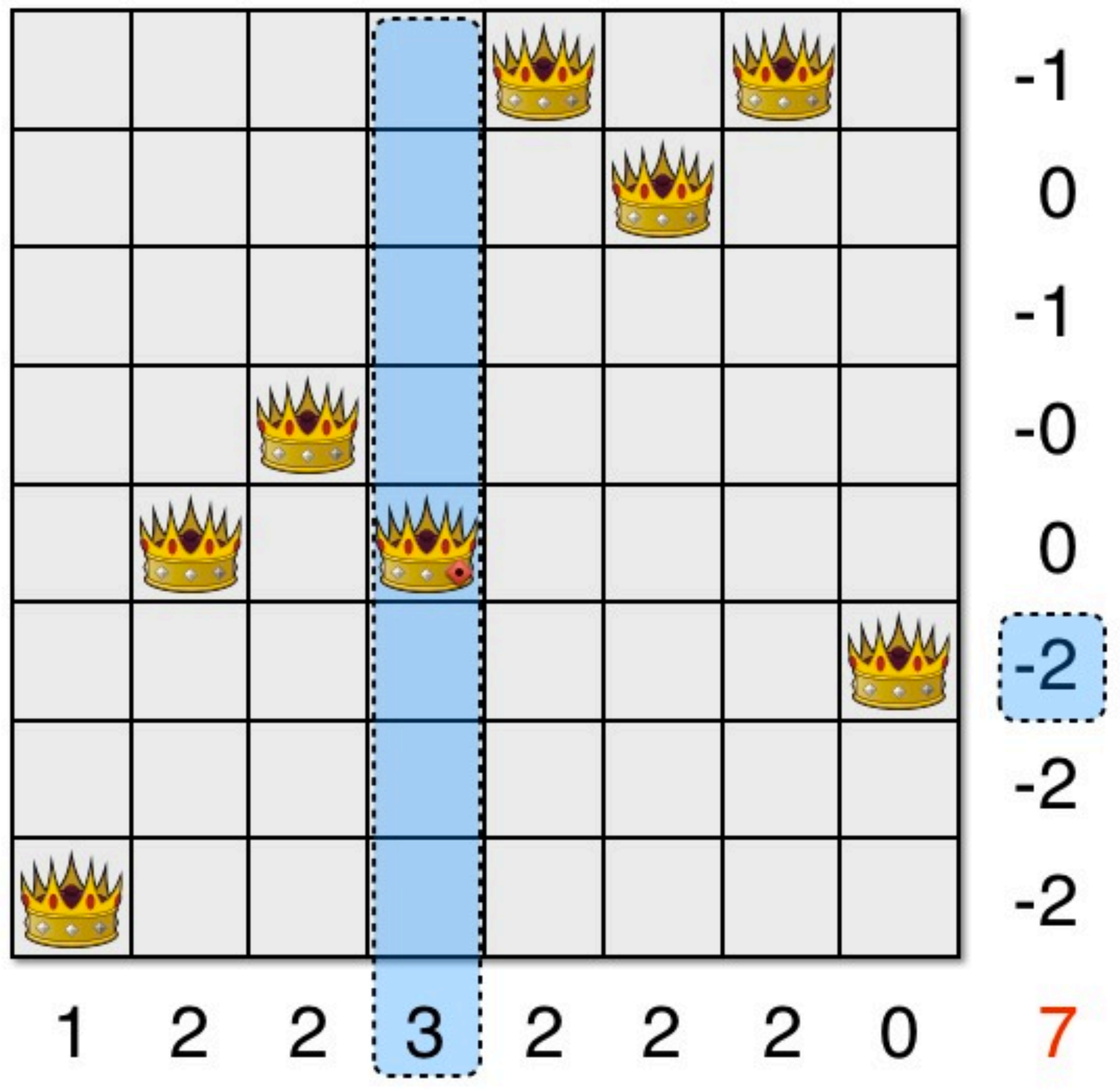

Multi-Stage Heuristics

- Max/Min-Confict
  - select the variable with the most violations
    - first stage: greedy
  - select the value with the fewest resulting violations
    - second stage: greedy

Multi-Stage Heuristics

- Max/Min-Confict
  - select the variable with the most violations
    - first stage: greedy
  - select the value with the fewest resulting violations
    - second stage: greedy
- Min-conflict heuristic
  - randomly select a variable with some violations
    - first-stage: randomized
  - select the value with the fewest resulting violations
    - second stage: greedy

Multi-Stage Heuristics

- What was the alternative?
  - N(s): { s[q ←v] | q in Queens & v in Rows }
    - s[q ←v] is the solution s where queens q is assigned to v;

Multi-Stage Heuristics

- What was the alternative?
  - N(s): { s[q ←v] I q in Queens & v in Rows }
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- Complexity
  - quadratic: all pairs (c,r) where c is a column and r is a row
  - O(n^2) where n is the number of queens

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  - quadratic: all pairs (c,r) where c is a column and r is a row
  - O(n^2) where n is the number of queens
- Complexity of min-conflict
  - O(n) where n is the number of queens

Multi-Stage in Car Sequencing

Slots	1	2	3	4	5	6	7	8	9	10	Demand
Class 1	🟦										1
Class 2		🟦									1
Class 3			🟦	🟦							2
Class 4					🟦	🟦					2
Class 5							🟦	🟦			2
Class 6									🟦	🟦	2

Options	1	2	3	4	5	Demand
Class 1	yes		yes	yes		1
Class 2				yes		1
Class 3		yes			yes	2
Class 4		yes		yes		2
Class 5	yes		yes			2
Class 6	yes	yes				2
Capacity	1/2	2/3	1/3	2/5	1/5	

Setup	1	2	3	4	5	6	7	8	9	10	Capacity	
Option 1	🟦						✴	✴	✴	✴	1/2	3
Option 2			✴	✴	✴	✴			🟦	🟦	2/3	2
Option 3	🟦						✴	✴			1/3	2
Option 4	✴	✴			✴	✴					2/5	2
Option 5			✴	✴							1/5	3

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Option 4	✖	✖			✖	✖					2/5	2
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  - consider all possible swaps
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Multi-Stage Heuristics

- Quadratic neighborhood
  - consider all possible swaps
  - quadratic in the size of the assembly line
- Multi-stage neighborhood
  - select a slot s whose car induces some violations
  - swap slot s with all other slots
  - linear in the size of the assembly line

Random Walks

- Randomization
  - select a neighbor at random

Random Walks

- Randomization
  - select a neighbor at random
- Decide whether to accept it
  - random improvement
  - Metropolis algorithm

Random Walks

- Randomization
  - select a neighbor at random
- Random improvement
- Random improvement search

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```
1. **function** S-RandomImprovement(N,s)
2.     **select** $n \in N$ with probability $1/\#N$;
3.     **if** $f(n) < f(s)$ **then**
4.         **return** $n$;
5.     **else**
6.         **return** $s$;
```

- Random improvement search

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```

- Random improvement search

```
1. **function** RandomImprovement(s)
2.     **return** LocalSearch($f$,$N$,L-All,S-RandomImprovement);
```

The Traveling Tournament

T-R	1	2	3	4	5	6	7	8	9	10
1	6	@2	4	3	@5	@4	@3	5	2	@6
2	5	1	@3	@6	4	3	6	@4	@1	@5
3	@4	5	2	@1	6	@2	1	@6	@5	4
4	3	6	@1	@5	@2	1	5	2	@6	@3
5	@2	@3	6	4	1	@6	@4	@1	3	2
6	@1	@4	@5	2	@3	5	@2	3	4	1

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4	3	6	@1	@5	@2	1	5	2	@6	@3
5	@2	@3	6	4	1	@6	@4	@1	3	2
6	@1	@4	@5	2	@3	5	@2	3	4	1

$$d_{12} + d_{21} + d_{15} + d_{54} + d_{43} + d_{31} + d_{16} + d_{61}$$

$+ \ldots +$

$$d_{61} + d_{14} + d_{45} + d_{56} + d_{63} + d_{36} + d_{62} + d_{26}$$

The Neighborhood

- A number of moves
  - swap homes
  - swap rounds
  - swap teams
  - partial swap rounds
  - partial swap teams

Swap Teams

T-R	1	2	3	4	5	6	7	8	9	10
1	6	@2	4	3	@5	@4	@3	5	2	@6
2	5	1	@3	@6	4	3	6	@4	@1	@5
3	@4	5	2	@1	6	@2	1	@6	@5	4
4	3	6	@1	@5	@2	1	5	2	@6	@3
5	@2	@3	6	4	1	@6	@4	@1	3	2
6	@1	@4	@5	2	@3	5	@2	3	4	1

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3	@4	5	2	@1	6	@2	1	@6	@5	4
4	3	6	@1	@5	2	1	5	@2	@6	@3
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6	@1	@4	@5	2	@3	5	@2	3	4	1

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5	@2	1	@3	@6	4	3	6	@4	@1	2
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3	@4	2	5	@1	6	@5	1	@6	@2	4
4	3	6	@1	@2	@5	1	2	5	@6	@3
5	@2	1	@3	@6	4	3	6	@4	@1	2
6	@1	@4	@2	5	@3	2	@5	3	4	1

Until Next Time