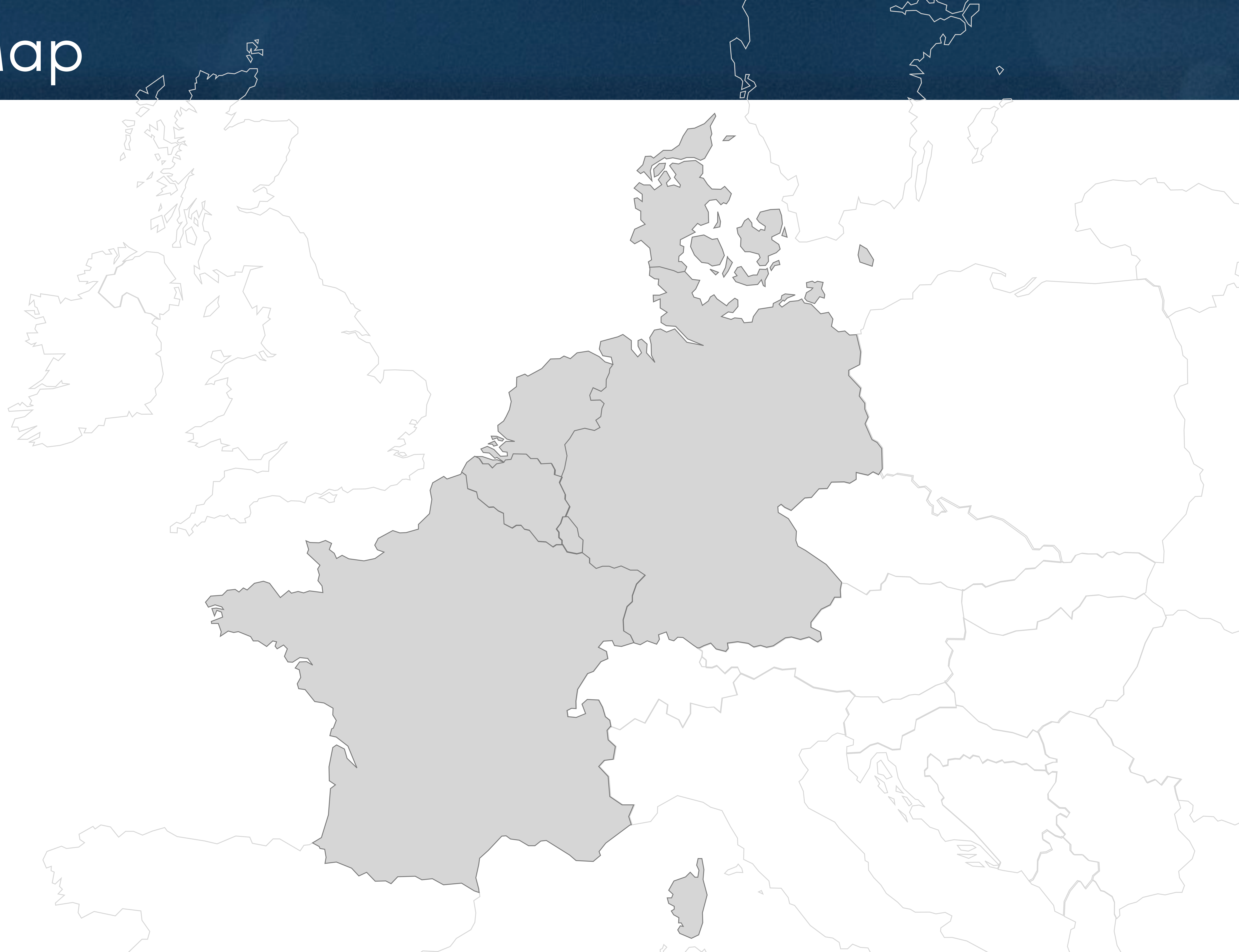
Discrete Optimization

Local Search: Part IV

Goals of the Lecture

- Local search
  - optimization under constraints
  - graph coloring

Coloring a Map



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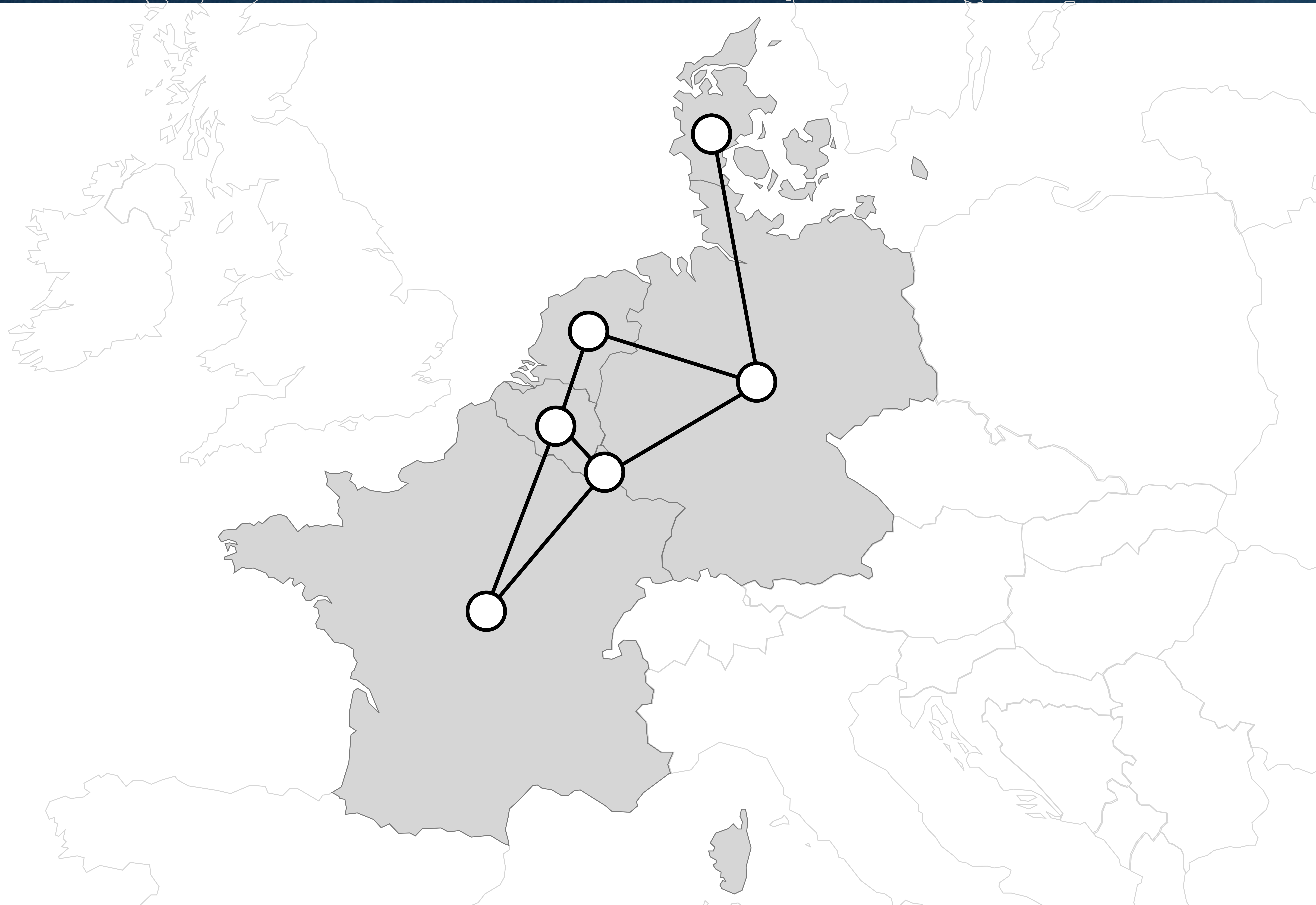

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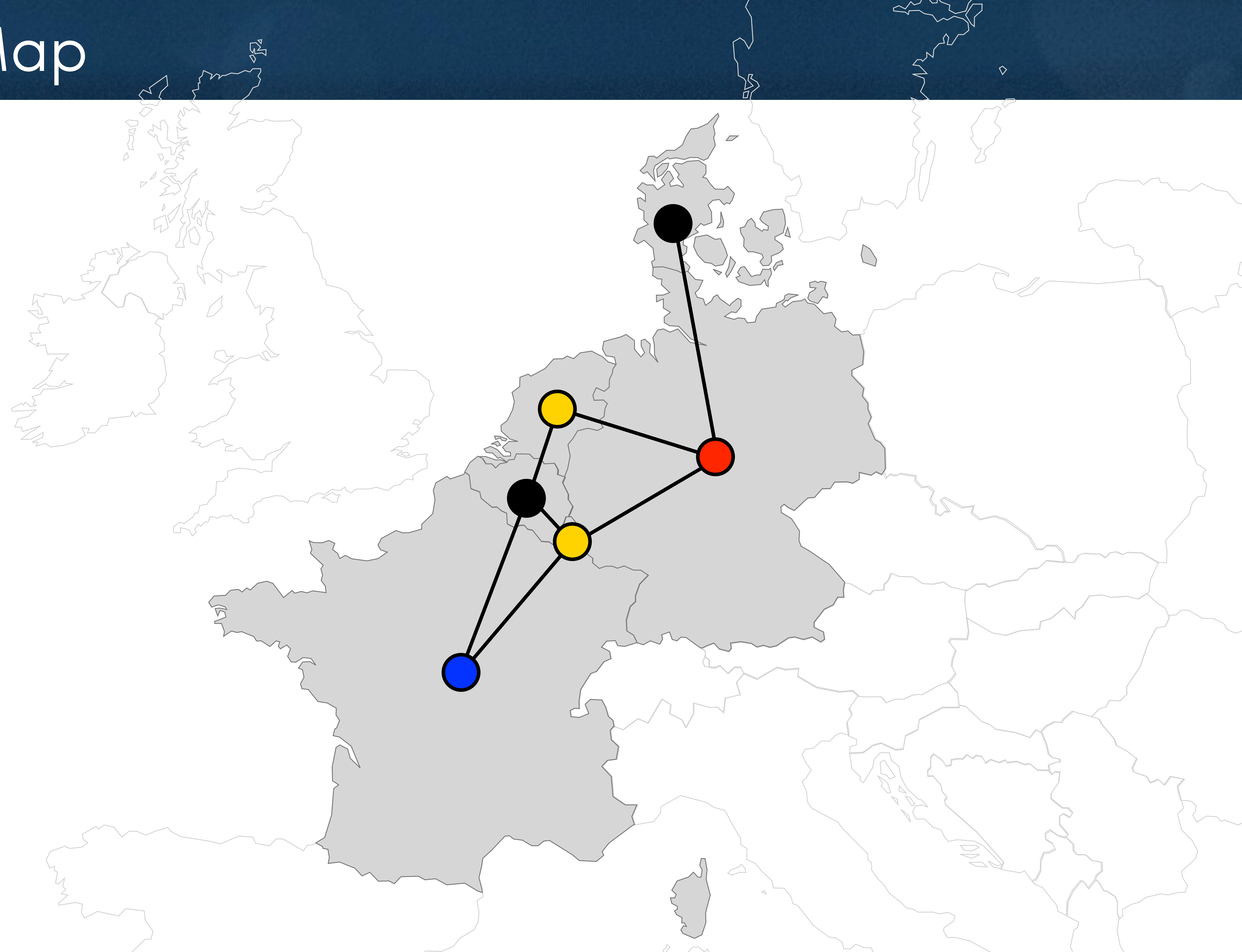
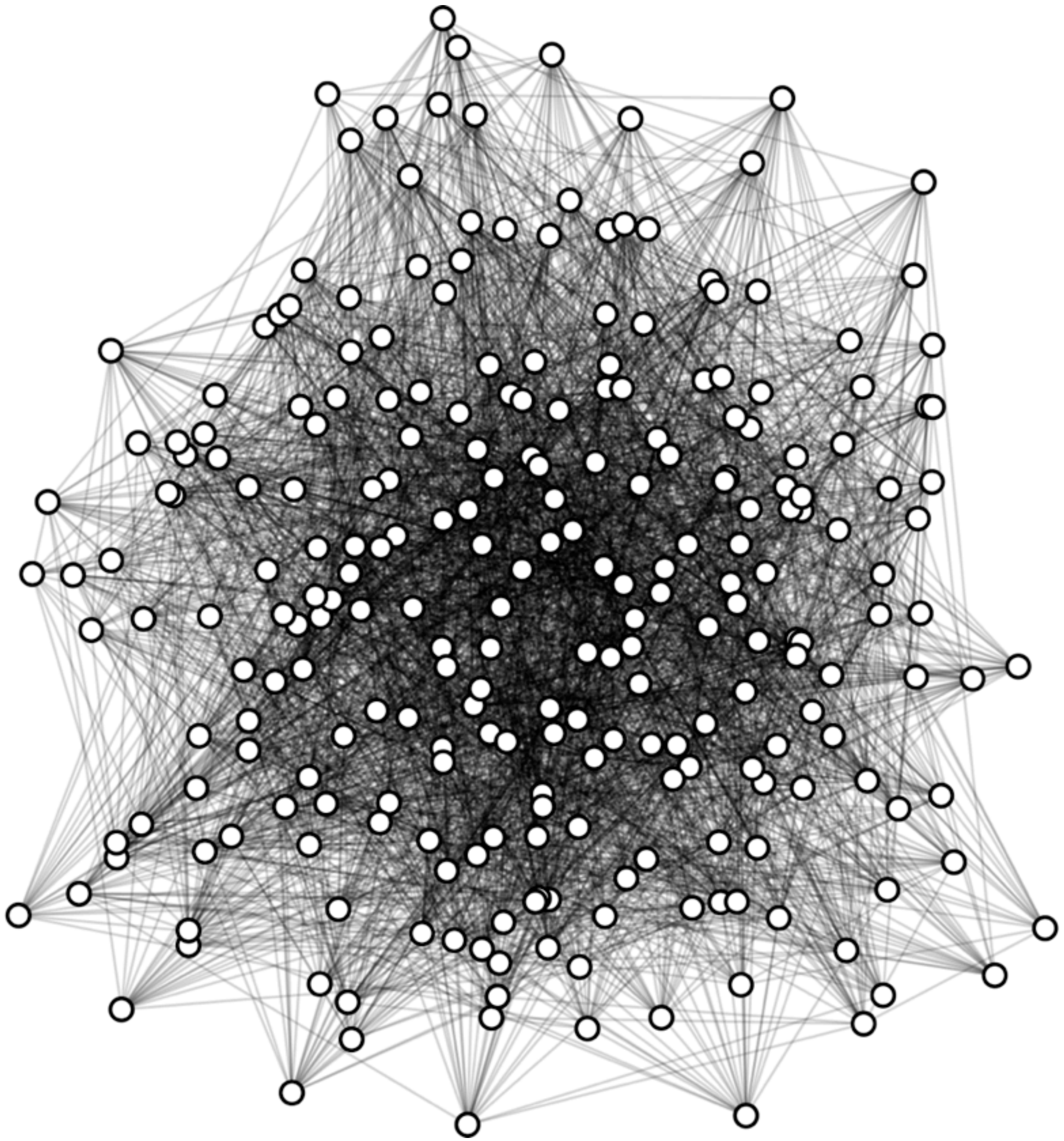


Graph Coloring



Graph Coloring

- Two aspects
  - optimization
    - reducing the number of colors
  - feasibility:
    - two adjacent vertices must be colored differently

Graph Coloring

- Two aspects
  - optimization
    - reducing the number of colors
  - feasibility:
    - two adjacent vertices must be colored differently
- How to combine them in local search?
  - sequence of feasibility problems
  - staying in the space of solutions
  - considering feasible and infeasible configurations

Optimization as Feasibility

- Sequence of feasibility problems
  - find an initial solution with k colors
    - greedy algorithms
  - remove one color, say k.
    - reassign randomly all vertices colored with k with a color in the range 1..k-1
  - find a feasible solution with k-1 colors
  - repeat

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  - repeat
- How to find a solution with k-1 colors
  - we have seen that in the first two lectures
  - just minimize the violations

Staying in the Feasible Space

- Neighborhood
  - change the color of a vertex

Staying in the Feasible Space

- Neighborhood
  - change the color of a vertex
- Objective function
  - minimizing the number of colors

Staying in the Feasible Space

- Neighborhood
  - change the color of a vertex
- Objective function
  - minimizing the number of colors
- How to guide the search?
  - changing the color of a vertex typically does not change the number of colors

Staying in the Feasible Space

- Color classes
  - C_i is the set of vertices colored with i

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- How to drive the search?
  - use a proxy as objective function
  - favor large color classes

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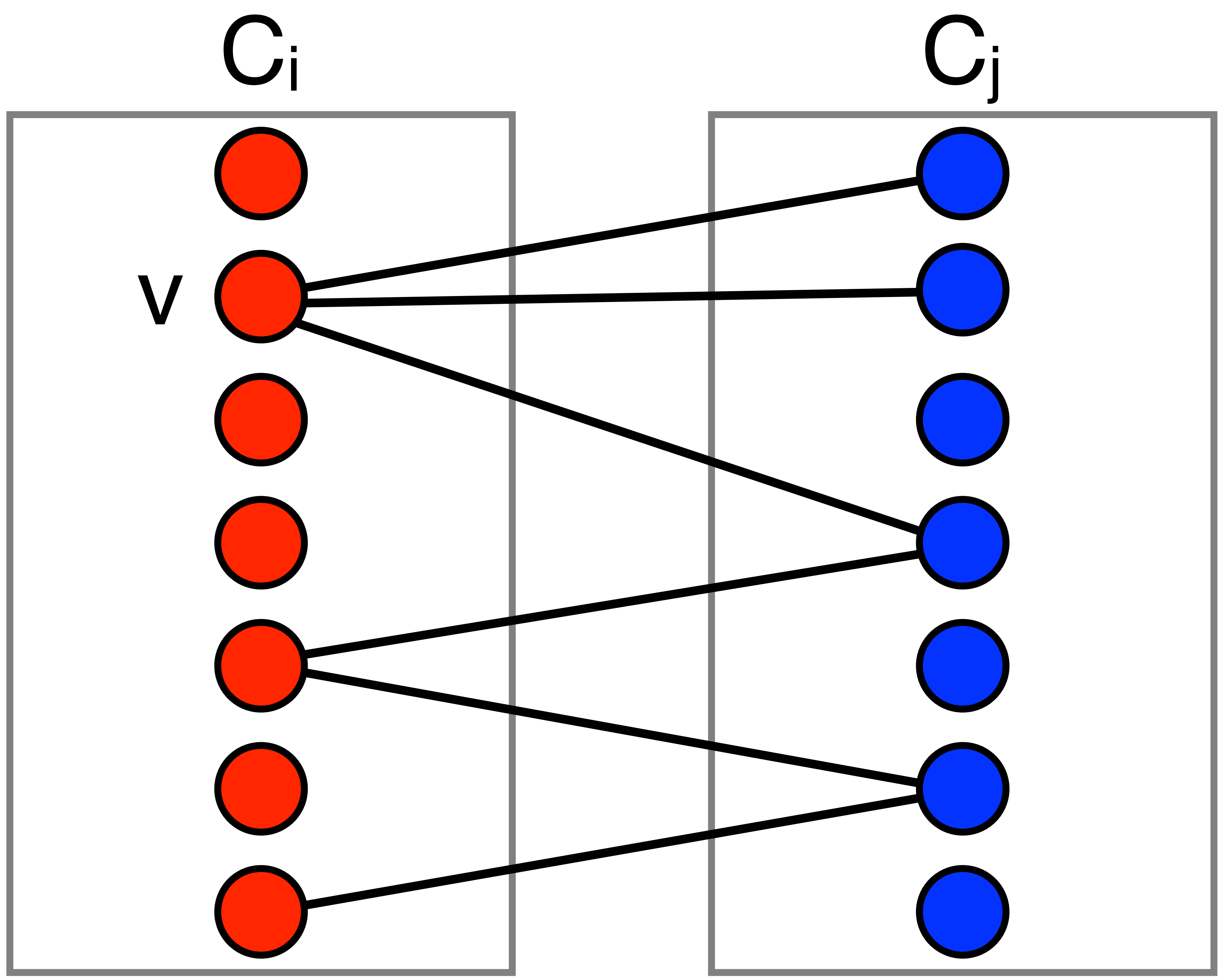
- Color classes
  - C_i is the set of vertices colored with i
- How to drive the search?
  - use a proxy as objective function
  - favor large color classes
- The objective function becomes

$$\text{maximize} \quad \sum_{i=1}^{n} |C_i|^2$$

Staying in the Feasible Space

- Richer neighborhoods
  - exploiting problem structure better
- Kemp Chains



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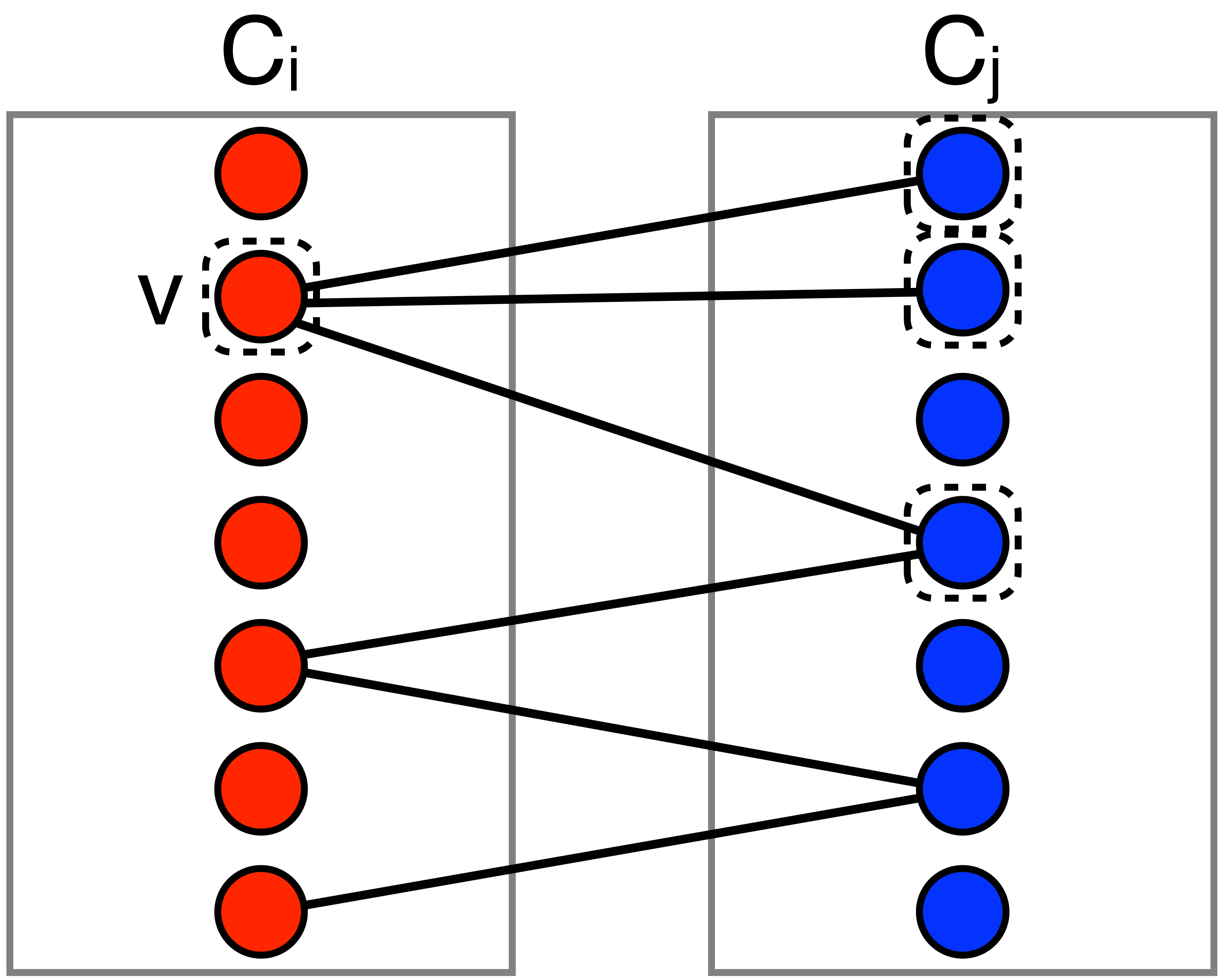

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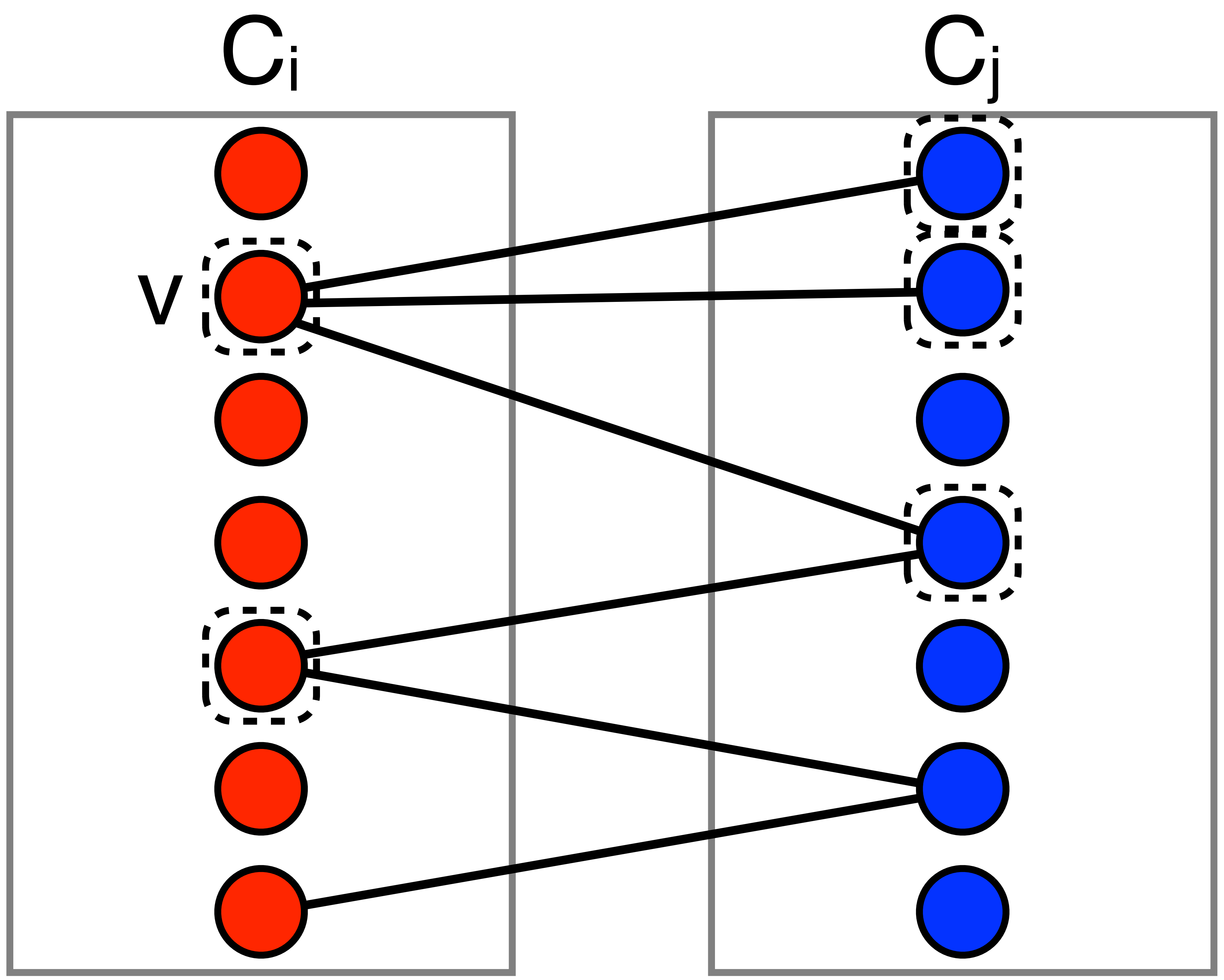

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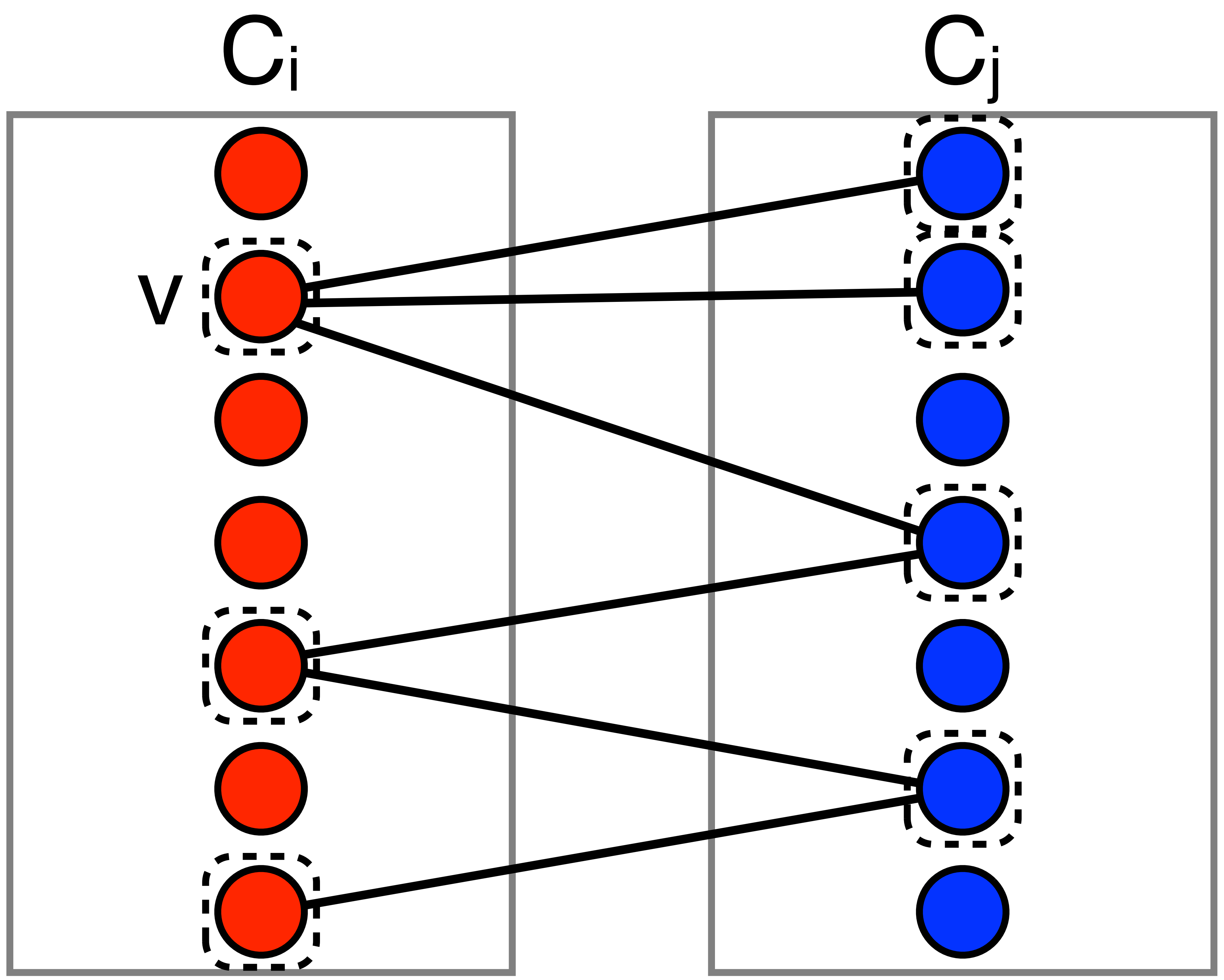

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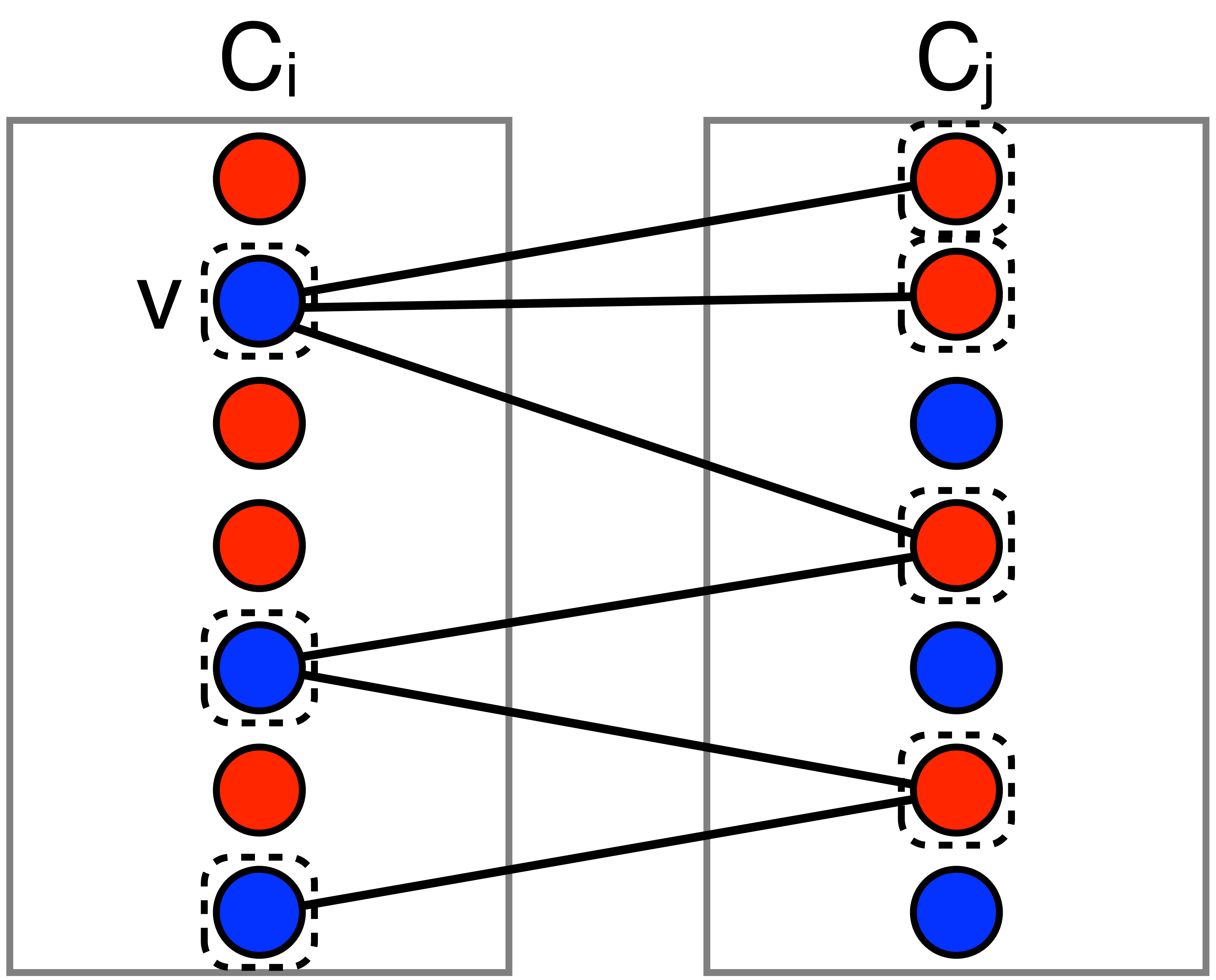

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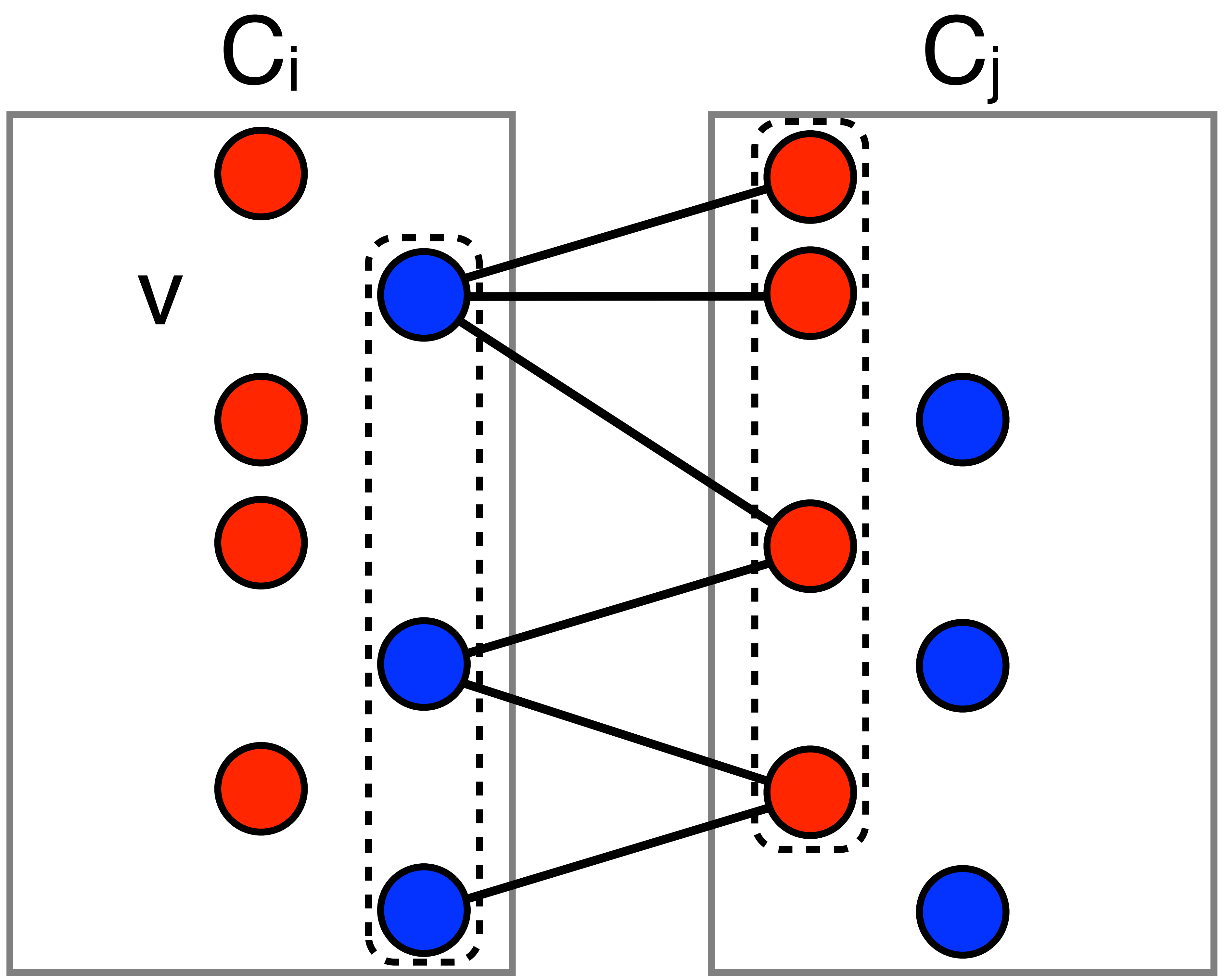

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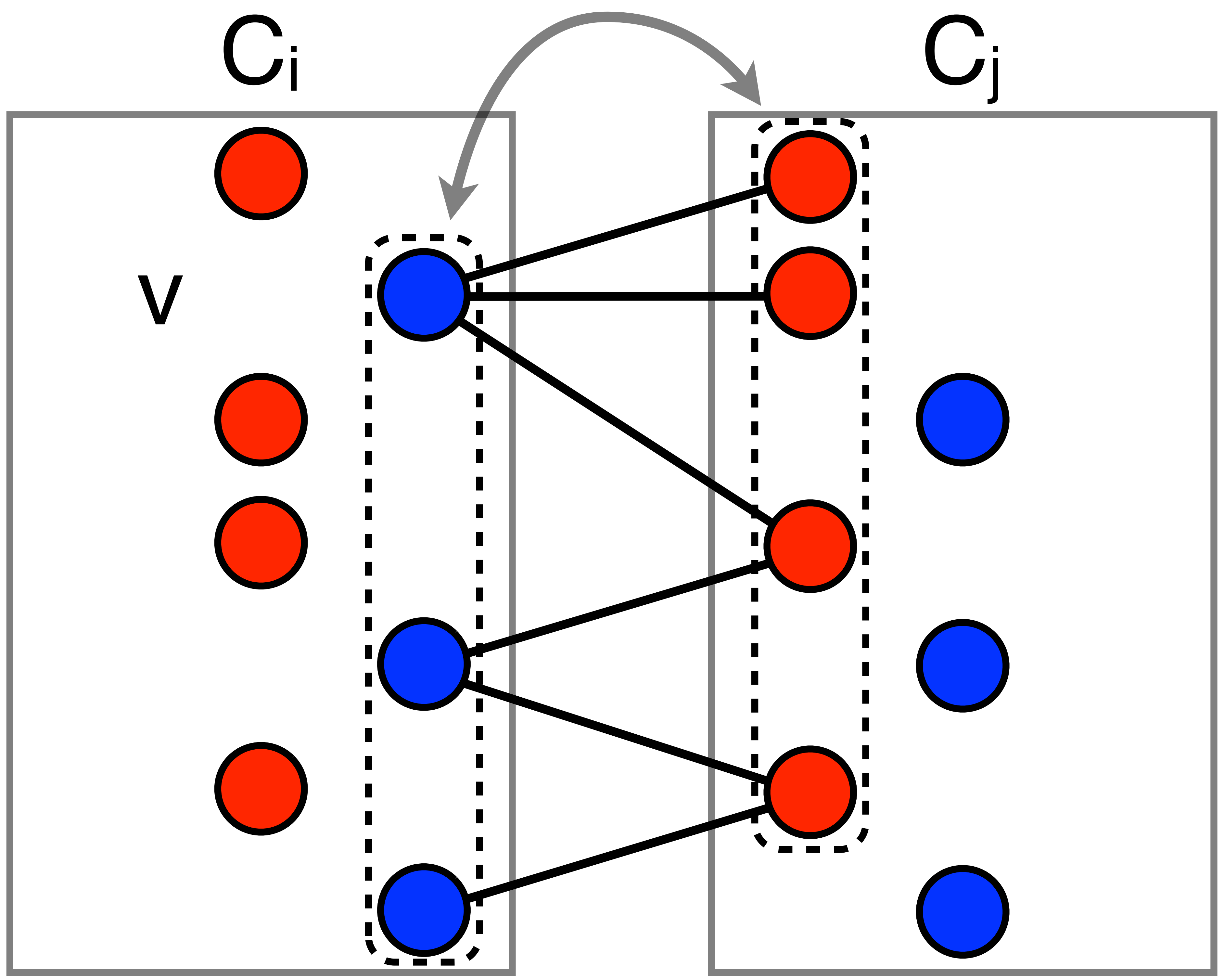

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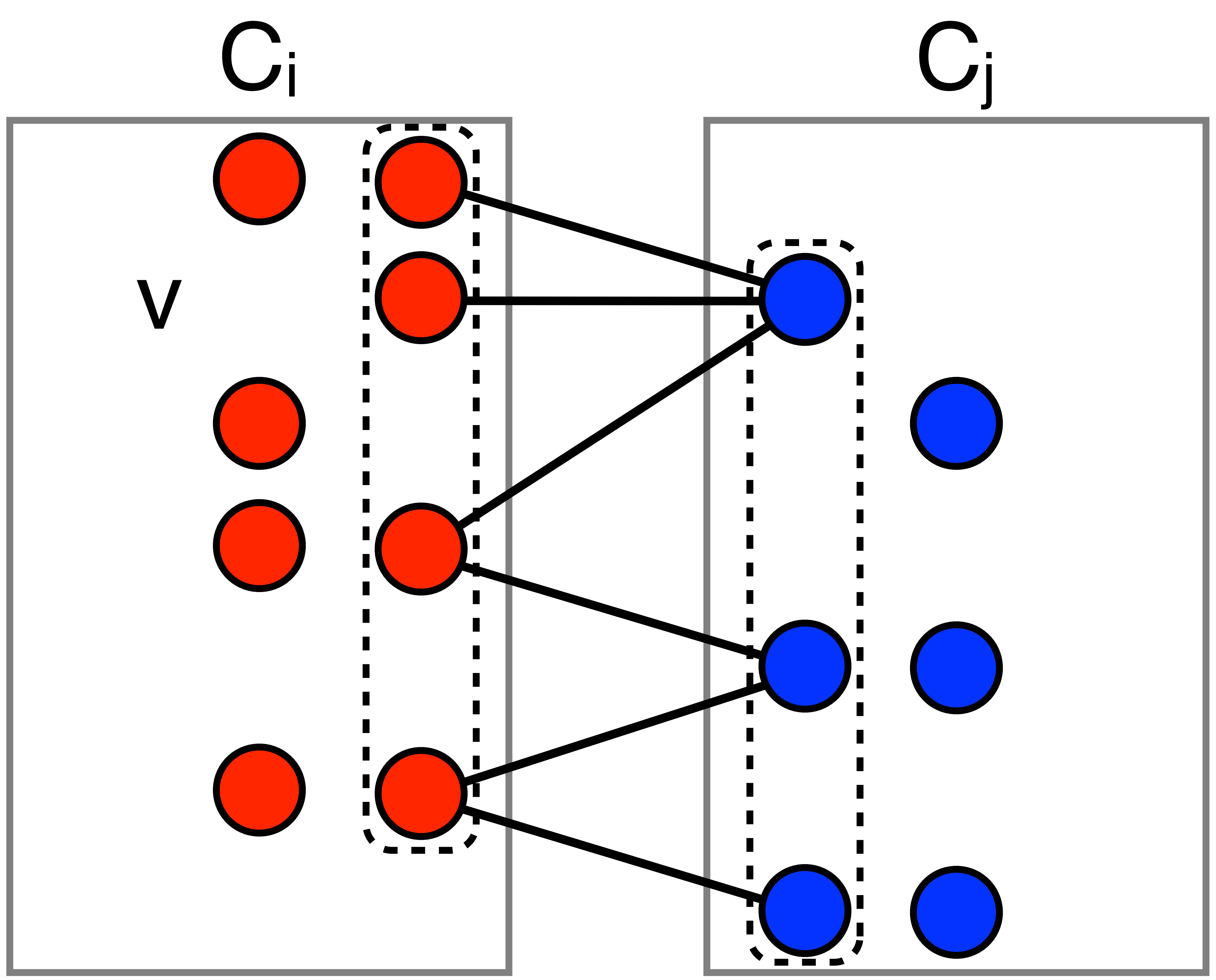

Exploring both Feasible and Infeasible Colorings

- Explore both feasible and infeasible colorings
  - the search must focus on reducing the number of colors and on ensuring feasibility.

Exploring both Feasible and Infeasible Colorings

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  - the search must focus on reducing the number of colors and on ensuring feasibility.
- How to combine optimization and feasibility
  - make sure that local optima are feasible
  - use an objective function that balances feasibility and optimality

$$\text{minimize } w_f \; f + w_o \; O$$

Exploring both Feasible and Infeasible Colorings

- Neighborhood
  - change the color of a vertex

Exploring both Feasible and Infeasible Colorings

- Neighborhood
  - change the color of a vertex
- Bad edges
  - a bad edge is an edge whose adjacent vertices have the same color
  - B_i is the set of bad edges between vertices colored with i

Exploring both Feasible and Infeasible Colorings

- Neighborhood
  - change the color of a vertex
- Decreasing the number of colors

$$\text{maximize} \quad \sum_{i=1}^{n} |C_i|^2$$

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- Removing violations

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- How to combine them?

The Combined Objective Function

- Neighborhood
  - change the color of a vertex
- Objective function

$$\text{minimize} \sum_{i=1}^{n} 2\,|B_i|\,|C_i| - \sum_{i=1}^{n} |C_i|^2$$

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- Why?

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- Why?

local minima of this objective are legal colorings

Local Minima are Legal Colorings

- Consider a coloring $C_1,\ldots,C_k$
  - assume that B_i is not empty
  - we show that this coloring is not a local minimum

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- How does it vary?

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- How does the objective vary?

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  - the left term decreases by

$$2|B_i||C_i| - 2(|B_i|-1)(|C_i|-1) = 2|B_i| + 2|C_i| - 2 \geq 2|C_i|$$

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  - Overall, the objective decreases by at least 2

Until Next Time