

Discrete Optimization

Constraint Programming: Part II

Goals of the Lecture

- ▶ Illustrating more complex constraint propagation
- ▶ Showing that constraints have dedicated algorithms

Computational Paradigm

► Branch and prune

– pruning

- reduce the search space as much as possible

– branching

- decompose the problem into subproblems and explore the subproblems

Computational Paradigm

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- reduce the search space as much as possible

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- decompose the problem into subproblems and explore the subproblems

► Pruning

- use constraints to remove, from the variable domains, values that cannot belong to any solution

Computational Paradigm

► Branch and prune

– pruning

- reduce the search space as much as possible

– branching

- decompose the problem into subproblems and explore the subproblems

► Pruning

- use constraints to remove, from the variable domains, values that cannot belong to any solution

► Branching

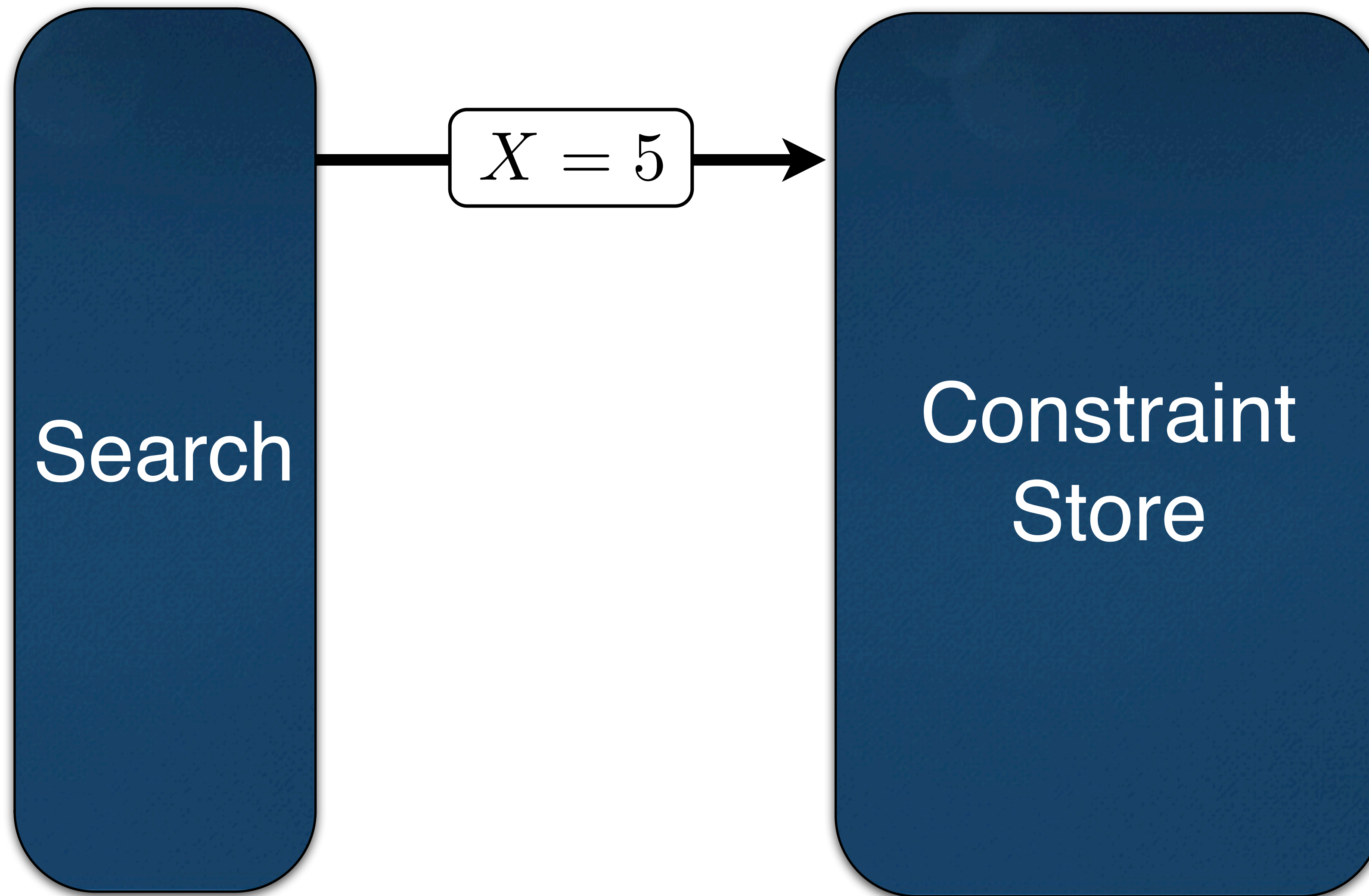
- e.g., try all the possible values of a variable until a solution is found or it can be proven that no solution exists

Computational Paradigm

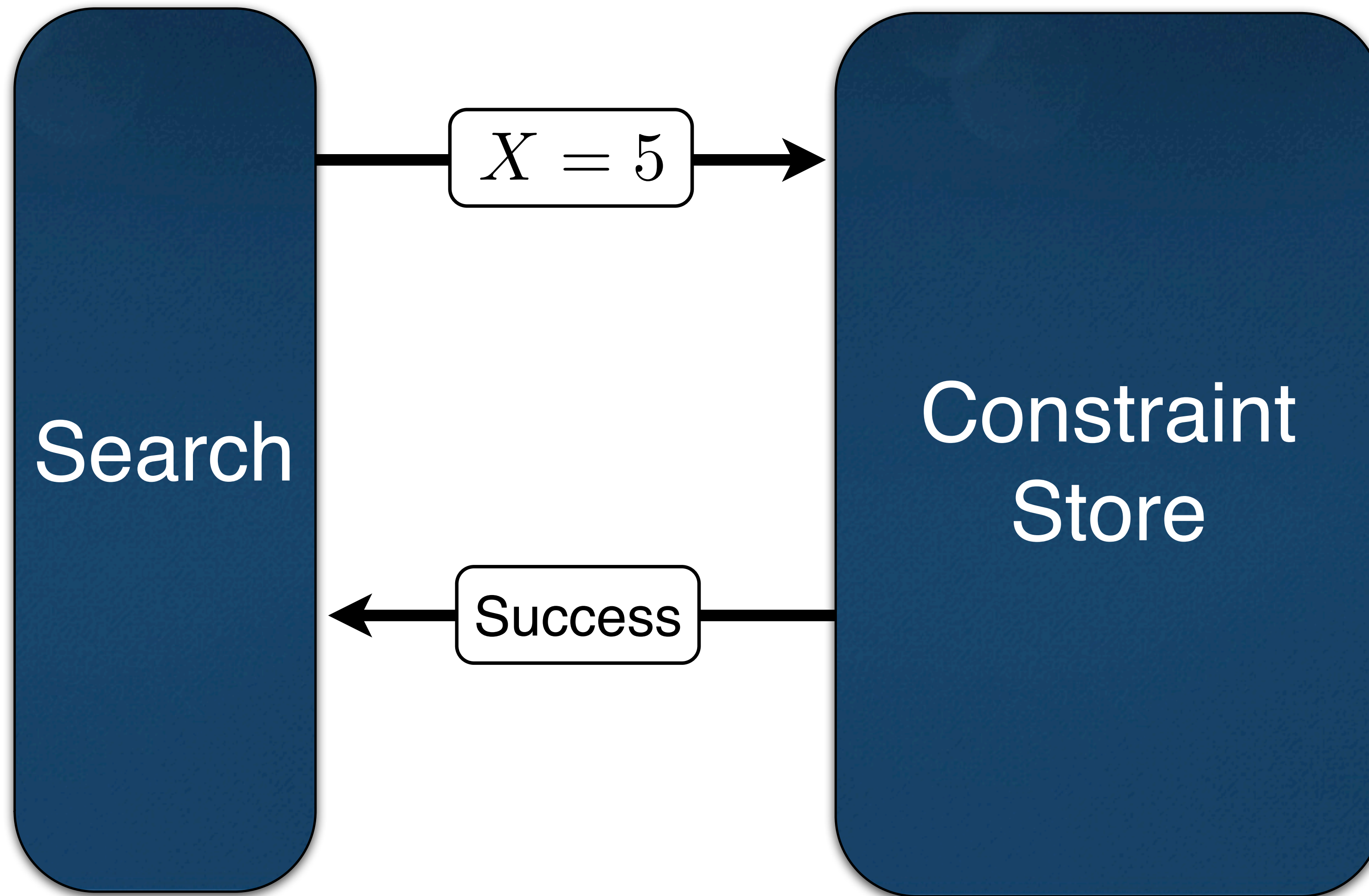
Search

Constraint
Store

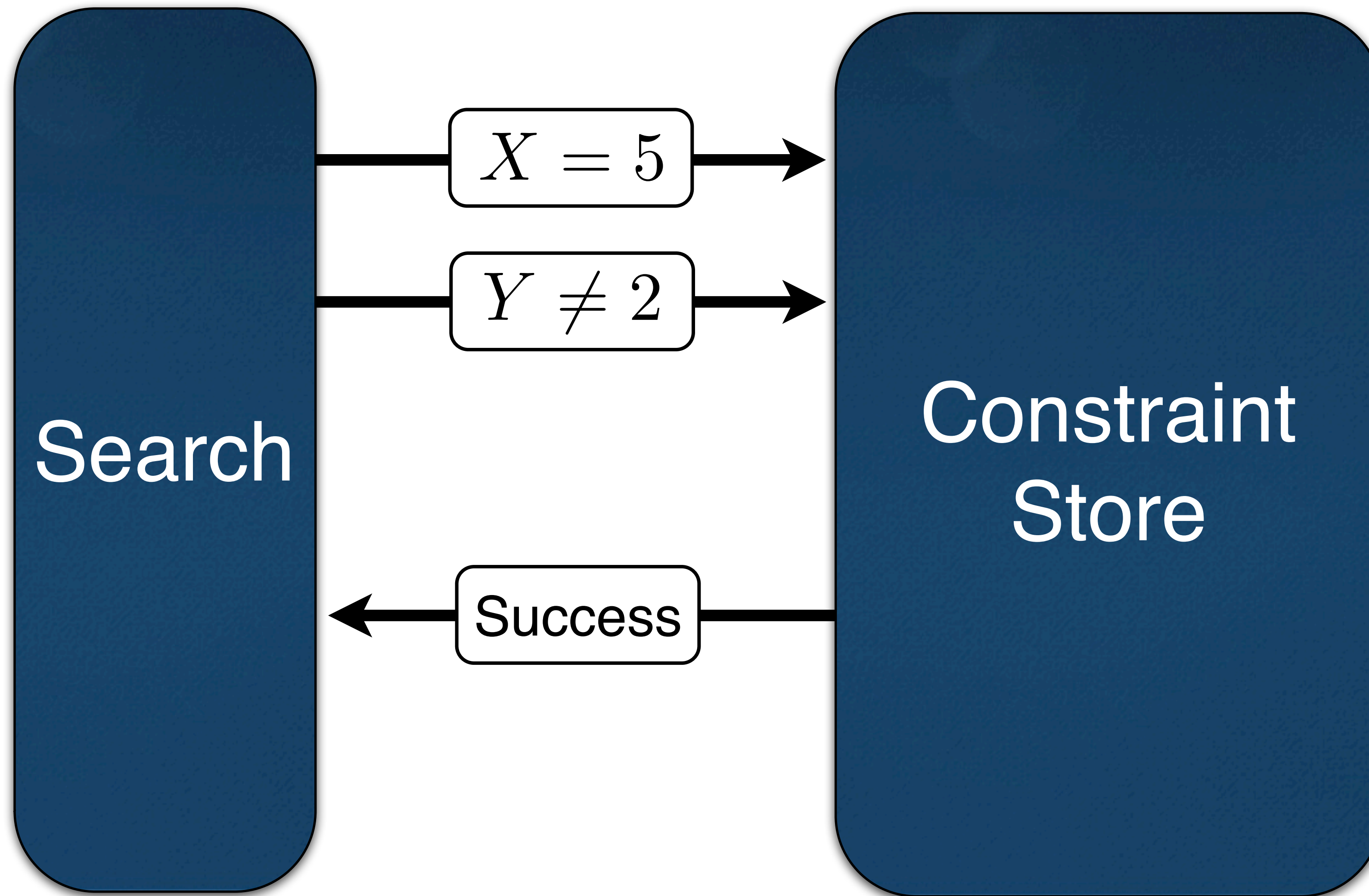
Computational Paradigm



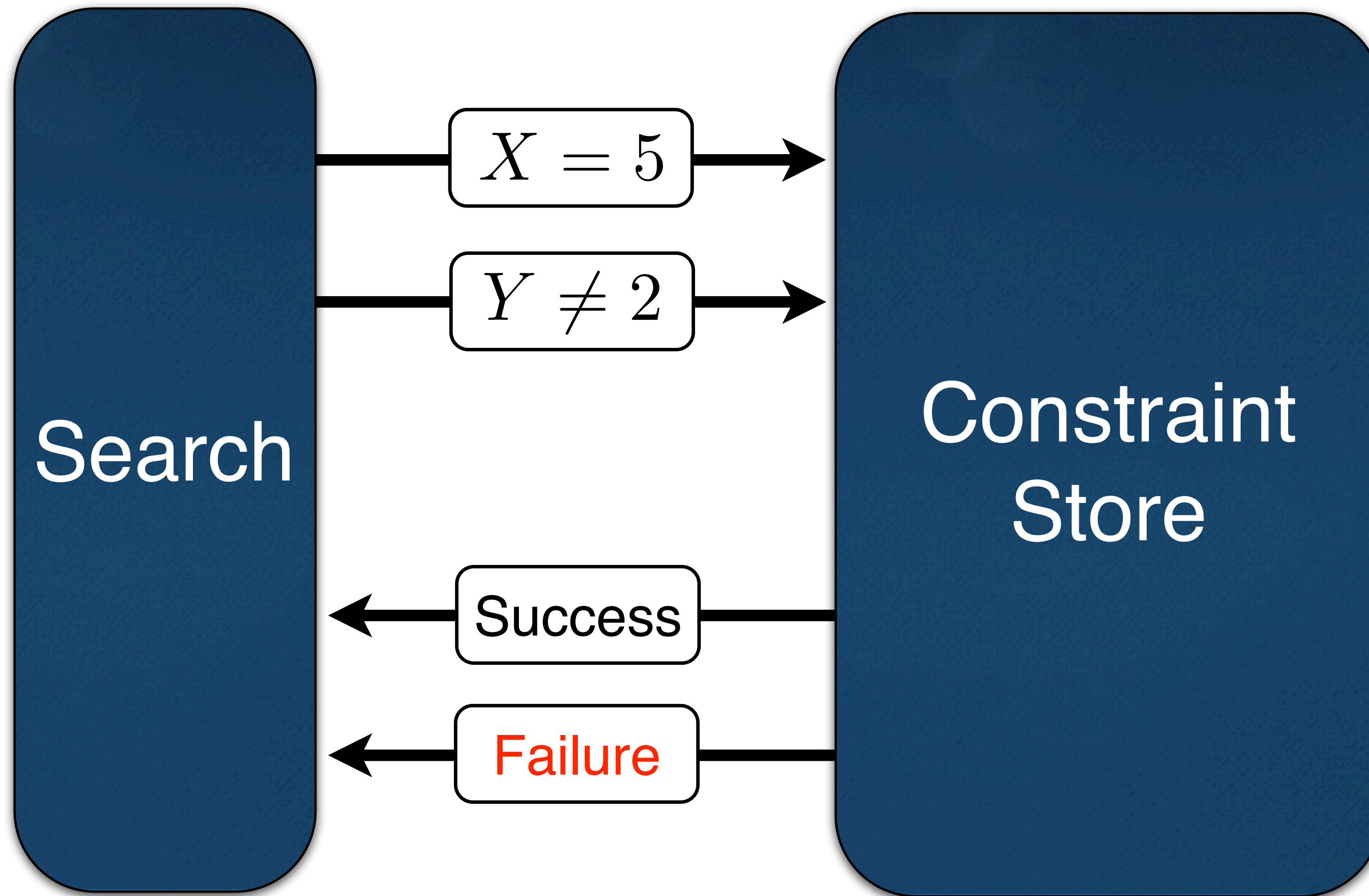
Computational Paradigm



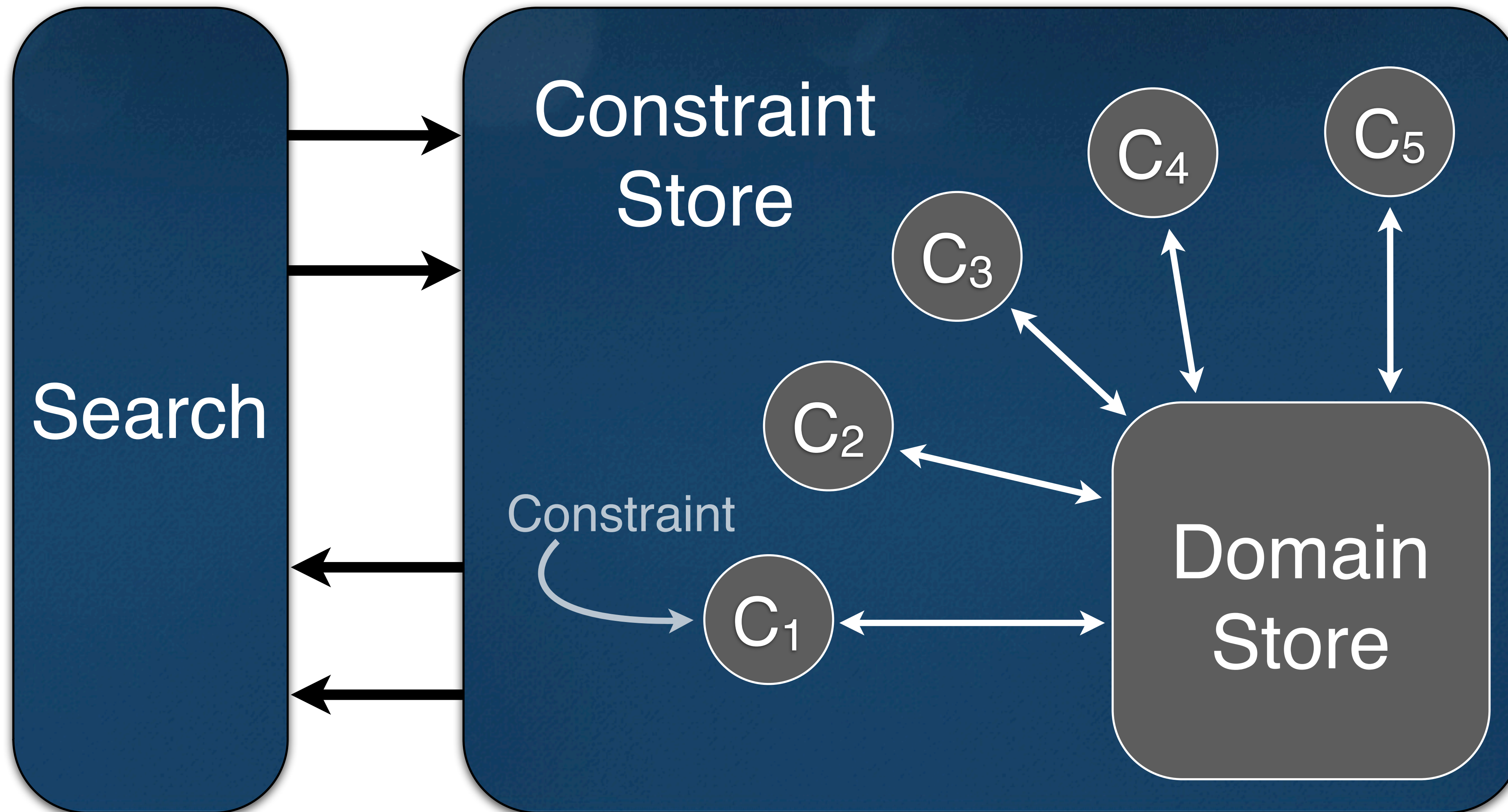
Computational Paradigm



Computational Paradigm



Computational Paradigm



Computational Paradigm

► What does a constraint do?

- feasibility checking
- pruning

Computational Paradigm

- ▶ What does a constraint do?
 - feasibility checking
 - pruning
- ▶ Feasibility checking
 - can a constraint be satisfied given the values in the domains of its variables

Computational Paradigm

- ▶ What does a constraint do?
 - feasibility checking
 - pruning
- ▶ Feasibility checking
 - can a constraint be satisfied given the values in the domains of its variables
- ▶ Pruning
 - if satisfiable, determine which values in the domains cannot be part of any solution

Computational Paradigm

- ▶ What does a constraint do?
 - feasibility checking
 - pruning
- ▶ Feasibility checking
 - can a constraint be satisfied given the values in the domains of its variables
- ▶ Pruning
 - if satisfiable, determine which values in the domains cannot be part of any solution
- ▶ The algorithms use dedicated algorithms for each constraint
 - they exploit the structure and properties of the constraint

Send More Money

► Specification

- assign different digits to letters to satisfy the addition

S	E	N	D		
+	M	O	R	E	
<hr/>					
=	M	O	N	E	Y

Send More Money

- Specification
 - assign different digits to letters to satisfy the addition
- Model to illustrate constraint propagation
 - no claim that this is a good model: it is not

S	E	N	D		
+	M	O	R	E	
<hr/>					
=	M	O	N	E	Y

Send More Money

- Basic modeling
 - add carries explicitly like in kindergarten

	C ₄	C ₃	C ₂	C ₁	
		S	E	N	D
	+	M	O	R	E
<hr/>					
=	M	O	N	E	Y

Send More Money

- ▶ Basic modeling
 - add carries explicitly like in kindergarten
- ▶ What are the decision variables
 - there is a variable for each letter to denote the value of the letters
 - there is a variable for each carry

	C_4	C_3	C_2	C_1	
		S	E	N	D
	+	M	O	R	E
<hr/>					
=	M	O	N	E	Y

Send More Money

```
enum Letters = { S, E, N, D, M, O, R, Y};
range Digits = 0..9;
var{int} value[Letters] in Digits;
var{int} carry[1..4] in 0..1;

solve {
  forall(i in Letters, j in Letters: i < j)
    value[i] ≠ value[j];
  value[S] ≠ 0;
  value[M] ≠ 0;
  carry[4] = value[M];
  carry[3] + value[S] + value[M] = value[O] + 10 * carry[4];
  carry[2] + value[E] + value[O] = value[N] + 10 * carry[3];
  carry[1] + value[N] + value[R] = value[E] + 10 * carry[2];
  value[D] + value[E] = value[Y] + 10 * carry[1];
}
```

	C ₄	C ₃	C ₂	C ₁	
		S	E	N	D
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$$\text{carry}[3] + \text{value}[S] + \text{value}[M] = \text{value}[O] + 10 * \text{carry}[4];$$

What is the Search Space?

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$$\text{carry}[3] + \text{value}[S] + \text{value}[M] = \text{value}[O] + 10 * \text{carry}[4];$$

What is the Search Space?

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$$\text{carry}[3] + \text{value}[S] + \text{value}[M] = \text{value}[0] + 10 * \text{carry}[4];$$

$$\text{carry}[3] + \text{value}[S] + 1 = \text{value}[0] + 10 * \text{carry}[4];$$

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$\in [3..11]$

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$$\in [3..11]$$
$$\in [0 + 2 + 1 .. 1 + 9 + 1]$$

What is the Search Space?

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$\in [3..11]$
 $\in [0 + 2 + 1 .. 1 + 9 + 1]$

What is the Search Space?

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$\in [3..11]$
 $\in [0 + 2 + 1 .. 1 + 9 + 1]$

What is the Search Space?

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$\in [3..11]$
 $\in [0 + 2 + 1 .. 1 + 9 + 1]$

What is the Search Space?

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$$\text{carry}[3] + \text{value}[S] + \text{value}[M] = \text{value}[0] + 10 * \text{carry}[4];$$

$$\text{carry}[3] + \text{value}[S] + 1 = \text{value}[0] + 10 * \text{carry}[4];$$

$\in [3..11]$
 $\in [0 + 2 + 1 .. 1 + 9 + 1]$

What is the Search Space?

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$$\text{carry}[3] + \text{value}[S] + \text{value}[M] = \text{value}[0] + 10 * \text{carry}[4];$$

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$\in [3..11]$
 $\in [0 + 2 + 1 .. 1 + 9 + 1]$

What is the Search Space?

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$$\text{carry}[3] + \text{value}[S] + 1 = \text{value}[0] + 10 * \text{carry}[4];$$

$\in [3..11]$
 $\in [0 + 2 + 1 .. 1 + 9 + 1]$

$\in [10..19]$

What is the Search Space?

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$$\text{carry}[3] + \text{value}[S] + \text{value}[M] = \text{value}[0] + 10 * \text{carry}[4];$$

$$\text{carry}[3] + \text{value}[S] + 1 = \text{value}[0] + 10 * \text{carry}[4];$$

$$\begin{aligned} &\in [3..11] \\ &\in [0 + 2 + 1 .. 1 + 9 + 1] \end{aligned}$$

$$[3..11] \cap [10..19] = [10..11]$$

What is the Search Space?

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$$\text{carry}[3] + \text{value}[S] + \text{value}[M] = \text{value}[0] + 10 * \text{carry}[4];$$

$$\text{carry}[3] + \text{value}[S] + 1 = \text{value}[0] + 10 * \text{carry}[4];$$

$$10 \leq \text{carry}[3] + \text{value}[S] + 1 \leq 11$$

$$\in [0 + 2 + 1 .. 1 + 9 + 1]$$

$$[3..11] \cap [10..19] = [10..11]$$

What is the Search Space?

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10 <= carry[3] + value[S] + 1 <= 11

What is the Search Space?

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$10 \leq \text{carry}[3] + \text{value}[S] + 1 \leq 11$

$10 - 1 \leq \text{carry}[3] + \text{value}[S] \leq 11 - 1$

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$$10 \leq \text{carry}[3] + \text{value}[S] + 1 \leq 11$$

$$10 - 1 \leq \text{carry}[3] + \text{value}[S] \leq 11 - 1$$

$$10 - 1 - \text{carry}[3] \leq \text{value}[S] \leq 11 - 1 - \text{carry}[3]$$

What is the Search Space?

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$$10 - 1 \leq \text{carry}[3] + \text{value}[S] \leq 11 - 1$$

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$$10 - 1 - \text{carry}[3] \leq \text{value}[S] \leq 11 - 1 - \text{carry}[3]$$

$$10 - 1 - 1 \leq \text{value}[S] \leq 11 - 1 - 0$$

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$$10 - 1 - \text{carry}[3] \leq \text{value}[S] \leq 11 - 1 - \text{carry}[3]$$

$$10 - 1 - 1 \leq \text{value}[S] \leq 11 - 1 - 0$$

$$8 \leq \text{value}[S] \leq 10$$

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$$10 \leq \text{carry}[3] + \text{value}[S] + 1 \leq 11$$

$$10 - 1 \leq \text{carry}[3] + \text{value}[S] \leq 11 - 1$$

$$10 - 1 - \text{carry}[3] \leq \text{value}[S] \leq 11 - 1 - \text{carry}[3]$$

$$10 - 1 - 1 \leq \text{value}[S] \leq 11 - 1 - 0$$

$$8 \leq \text{value}[S] \leq 10$$

What is the Search Space?

	0	1	2	3	4	5	6	7	8	9
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What is the Search Space?

	0	1	2	3	4	5	6	7	8	9
S										
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Y										
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$$\text{carry}[3] + \text{value}[S] + 1 = \text{value}[0] + 10 * \text{carry}[4];$$

$\in [10..11]$

What is the Search Space?

	0	1	2	3	4	5	6	7	8	9
S										
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D										
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$$\text{carry}[3] + \text{value}[S] + 1 = \text{value}[0] + 10 * \text{carry}[4];$$

$$10 \leq \text{value}[0] + 10 * \text{carry}[4] \leq 11$$

What is the Search Space?

	0	1	2	3	4	5	6	7	8	9
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C ₄										
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$$\text{carry}[3] + \text{value}[S] + 1 = \text{value}[0] + 10 * \text{carry}[4];$$

$$10 \leq \text{value}[0] + 10 * \text{carry}[4] \leq 11$$

$\in [10..11]$

What is the Search Space?

	0	1	2	3	4	5	6	7	8	9
S										
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D										
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R										
Y										
C ₄										
C ₃										
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C ₁										

$$\text{carry}[3] + \text{value}[S] + 1 = \text{value}[0] + 10 * \text{carry}[4];$$

$$10 \leq \text{value}[0] + 10 * \text{carry}[4] \leq 11$$

$$10 \leq \text{value}[0] + 10 * 1 \leq 11$$

What is the Search Space?

	0	1	2	3	4	5	6	7	8	9
S										
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Y										
C ₄										
C ₃										
C ₂										
C ₁										

$$\text{carry}[3] + \text{value}[S] + 1 = \text{value}[0] + 10 * \text{carry}[4];$$

$$10 \leq \text{value}[0] + 10 * \text{carry}[4] \leq 11$$

$$10 \leq \text{value}[0] + 10 * 1 \leq 11$$

$$0 \leq \text{value}[0] \leq 1$$

What is the Search Space?

	0	1	2	3	4	5	6	7	8	9
S										
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D										
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Y										
C ₄										
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C ₂										
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$$\text{carry}[3] + \text{value}[S] + 1 = \text{value}[0] + 10 * \text{carry}[4];$$

$$10 \leq \text{value}[0] + 10 * \text{carry}[4] \leq 11$$

$$10 \leq \text{value}[0] + 10 * 1 \leq 11$$

$$0 \leq \text{value}[0] \leq 1$$

What is the Search Space?

	0	1	2	3	4	5	6	7	8	9
S										
E										
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C ₄										
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$$\text{carry}[3] + \text{value}[S] + 1 = \text{value}[0] + 10 * \text{carry}[4];$$

$$10 \leq \text{value}[0] + 10 * \text{carry}[4] \leq 11$$

$$10 \leq \text{value}[0] + 10 * 1 \leq 11$$

$$0 \leq \text{value}[0] \leq 1$$

What is the Search Space?

	0	1	2	3	4	5	6	7	8	9
S										
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C ₄										
C ₃										
C ₂										
C ₁										

$carry[3] + value[S] + 1 = value[0] + 10 * carry[4];$

$10 \leq value[0] + 10 * carry[4] \leq 11$

$10 \leq value[0] + 10 * 1 \leq 11$

$0 \leq value[0] \leq 1$

forall(i in Letters, j in Letters)
value[i] ≠ value[j];

What is the Search Space?

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										
C ₄										
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$$\text{carry}[3] + \text{value}[S] + 1 = \text{value}[0] + 10 * \text{carry}[4];$$

$$10 \leq \text{value}[0] + 10 * \text{carry}[4] \leq 11$$

$$10 \leq \text{value}[0] + 10 * 1 \leq 11$$

$$0 \leq \text{value}[0] \leq 1$$

forall(i in Letters, j in Letters)
value[i] ≠ value[j];

Send More Money

```
enum Letters = { S, E, N, D, M, O, R, Y};
range Digits = 0..9;
var{int} value[Letters] in Digits;
var{int} carry[1..4] in 0..1;

solve {
  forall(i in Letters, j in Letters: i < j)
    value[i] ≠ value[j];
  value[S] ≠ 0;
  value[M] ≠ 0;
  carry[4] = value[M];
  carry[3] + value[S] + value[M] = value[O] + 10 * carry[4];
  carry[2] + value[E] + value[O] = value[N] + 10 * carry[3];
  carry[1] + value[N] + value[R] = value[E] + 10 * carry[2];
  value[D] + value[E] = value[Y] + 10 * carry[1];
}
```

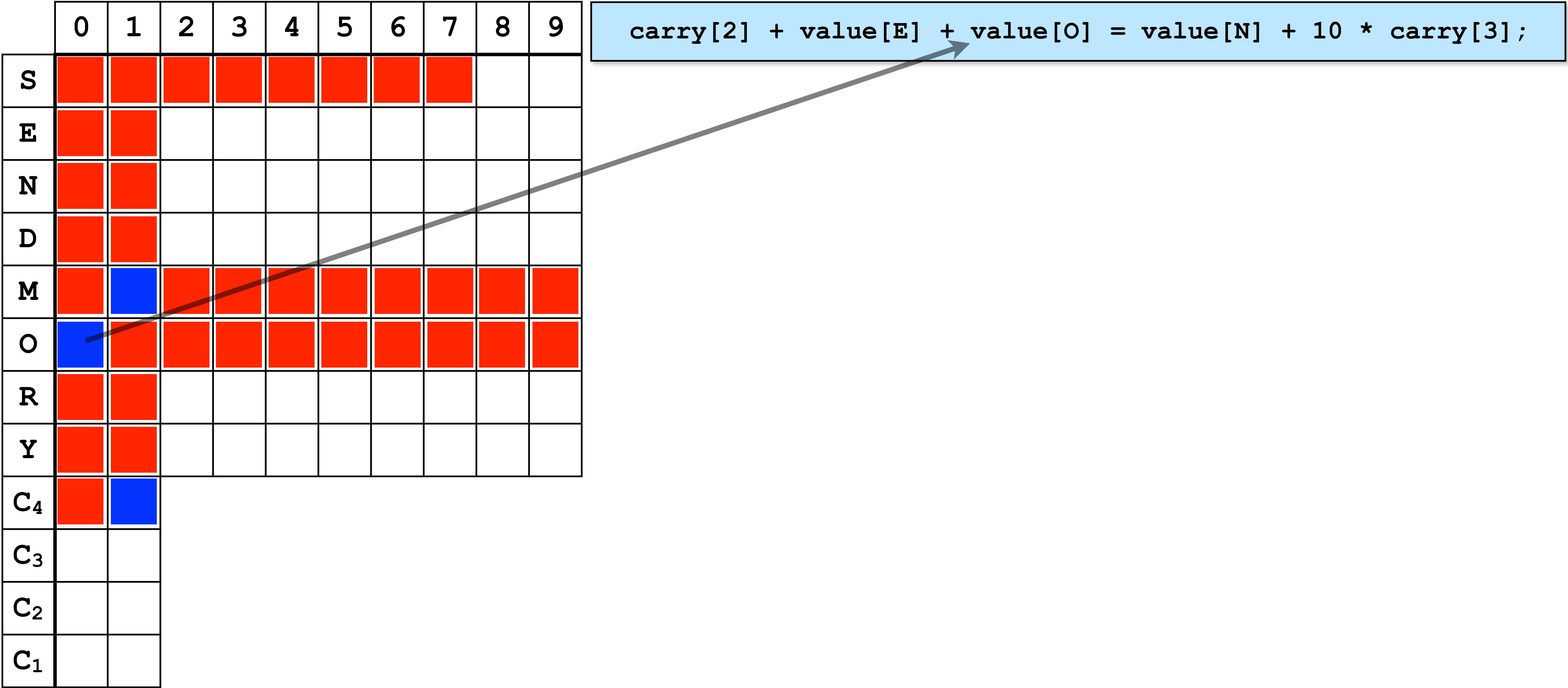
	C ₄	C ₃	C ₂	C ₁	
		S	E	N	D
+	M	O	R	E	
<hr/>					
=	M	O	N	E	Y

What is the Search Space?

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										
C ₄										
C ₃										
C ₂										
C ₁										

carry[2] + value[E] + value[0] = value[N] + 10 * carry[3];

What is the Search Space?



What is the Search Space?

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										
C ₄										
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C ₁										

$$\text{carry}[2] + \text{value}[\text{E}] + \text{value}[\text{O}] = \text{value}[\text{N}] + 10 * \text{carry}[3];$$

$$\text{carry}[2] + \text{value}[\text{E}] + 0 = \text{value}[\text{N}] + 10 * \text{carry}[3];$$

What is the Search Space?

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
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$$\text{carry}[2] + \text{value}[\text{E}] + \text{value}[\text{O}] = \text{value}[\text{N}] + 10 * \text{carry}[3];$$

$$\text{carry}[2] + \text{value}[\text{E}] + 0 = \text{value}[\text{N}] + 10 * \text{carry}[3];$$

$\in [2..10]$

What is the Search Space?

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
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C ₄										
C ₃										
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$$\text{carry}[2] + \text{value}[\text{E}] + \text{value}[\text{O}] = \text{value}[\text{N}] + 10 * \text{carry}[3];$$

$$\text{carry}[2] + \text{value}[\text{E}] + 0 = \text{value}[\text{N}] + 10 * \text{carry}[3];$$

$$\in [2..10]$$

$$2 \leq \text{value}[\text{N}] + 10 * \text{carry}[3] \leq 10$$

What is the Search Space?

	0	1	2	3	4	5	6	7	8	9
S										
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2 <= value[N] + 10 * carry[3] <= 10

What is the Search Space?

	0	1	2	3	4	5	6	7	8	9
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$$2 \leq \text{value}[N] + 10 * \text{carry}[3] \leq 10$$

$$2 - \text{value}[N] \leq 10 * \text{carry}[3] \leq 10 - \text{value}[N]$$

What is the Search Space?

	0	1	2	3	4	5	6	7	8	9
S										
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$$2 \leq \text{value}[N] + 10 * \text{carry}[3] \leq 10$$

$$2 - \text{value}[N] \leq 10 * \text{carry}[3] \leq 10 - \text{value}[N]$$

What is the Search Space?

	0	1	2	3	4	5	6	7	8	9
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$$2 \leq \text{value}[N] + 10 * \text{carry}[3] \leq 10$$

$$2 - \text{value}[N] \leq 10 * \text{carry}[3] \leq 10 + \text{value}[N]$$

What is the Search Space?

	0	1	2	3	4	5	6	7	8	9
S										
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$$2 \leq \text{value}[N] + 10 * \text{carry}[3] \leq 10$$

$$2 - \text{value}[N] \leq 10 * \text{carry}[3] \leq 10 - \text{value}[N]$$

$$-7 \leq 10 * \text{carry}[3] \leq 8$$

What is the Search Space?

	0	1	2	3	4	5	6	7	8	9
S										
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$$2 \leq \text{value}[N] + 10 * \text{carry}[3] \leq 10$$

$$2 - \text{value}[N] \leq 10 * \text{carry}[3] \leq 10 - \text{value}[N]$$

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$$2 \leq \text{value}[N] + 10 * \text{carry}[3] \leq 10$$

$$2 - \text{value}[N] \leq 10 * \text{carry}[3] \leq 10 - \text{value}[N]$$

$$-7 \leq 10 * \text{carry}[3] \leq 8$$

What is the Search Space?

	0	1	2	3	4	5	6	7	8	9
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$$2 \leq \text{value}[N] + 10 * \text{carry}[3] \leq 10$$

$$2 - \text{value}[N] \leq 10 * \text{carry}[3] \leq 10 - \text{value}[N]$$

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What is the Search Space?

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$$2 \leq \text{value}[N] + 10 * \text{carry}[3] \leq 10$$

$$2 - \text{value}[N] \leq 10 * \text{carry}[3] \leq 10 - \text{value}[N]$$

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What is the Search Space?

	0	1	2	3	4	5	6	7	8	9
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What is the Search Space?

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```
enum Letters = { S, E, N, D, M, O, R, Y};
range Digits = 0..9;
var{int} value[Letters] in Digits;
var{int} carry[1..4] in 0..1;

solve {
  forall(i in Letters, j in Letters: i < j)
    value[i] ≠ value[j];
  value[S] ≠ 0;
  value[M] ≠ 0;
  carry[4] = value[M];
  carry[3] + value[S] + value[M] = value[O] + 10 * carry[4];
  carry[2] + value[E] + value[O] = value[N] + 10 * carry[3];
  carry[1] + value[N] + value[R] = value[E] + 10 * carry[2];
  value[D] + value[E] = value[Y] + 10 * carry[1];
}
```

What is the Search Space?

	0	1	2	3	4	5	6	7	8	9
S										
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  carry[3] + value[S] + value[M] = value[O] + 10 * carry[4];
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```

What is the Search Space?

	0	1	2	3	4	5	6	7	8	9
S										
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var{int} value[Letters] in Digits;
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solve {
  forall(i in Letters, j in Letters: i < j)
    value[i] ≠ value[j];
  value[S] ≠ 0;
  value[M] ≠ 0;
  carry[4] = value[M];
  carry[3] ← value[S] + value[M] = value[O] + 10 * carry[4];
  carry[2] + value[E] + value[O] = value[N] + 10 * carry[3];
  carry[1] + value[N] + value[R] = value[E] + 10 * carry[2];
  value[D] + value[E] = value[Y] + 10 * carry[1];
}
```


Computational Paradigm

- ▶ The propagation engine
 - this is the core of any constraint-programming solver
 - a simple (fixpoint) algorithm

```
propagate()  
{  
  repeat  
    select a constraint c;  
    if c is infeasible given the domain store then  
      return failure;  
    else  
      apply the pruning algorithm associated with c;  
      until no constraint can remove any value from the  
      domain of its variables;  
    return success;  
}
```

What is the Search Space?

	0	1	2	3	4	5	6	7	8	9
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```
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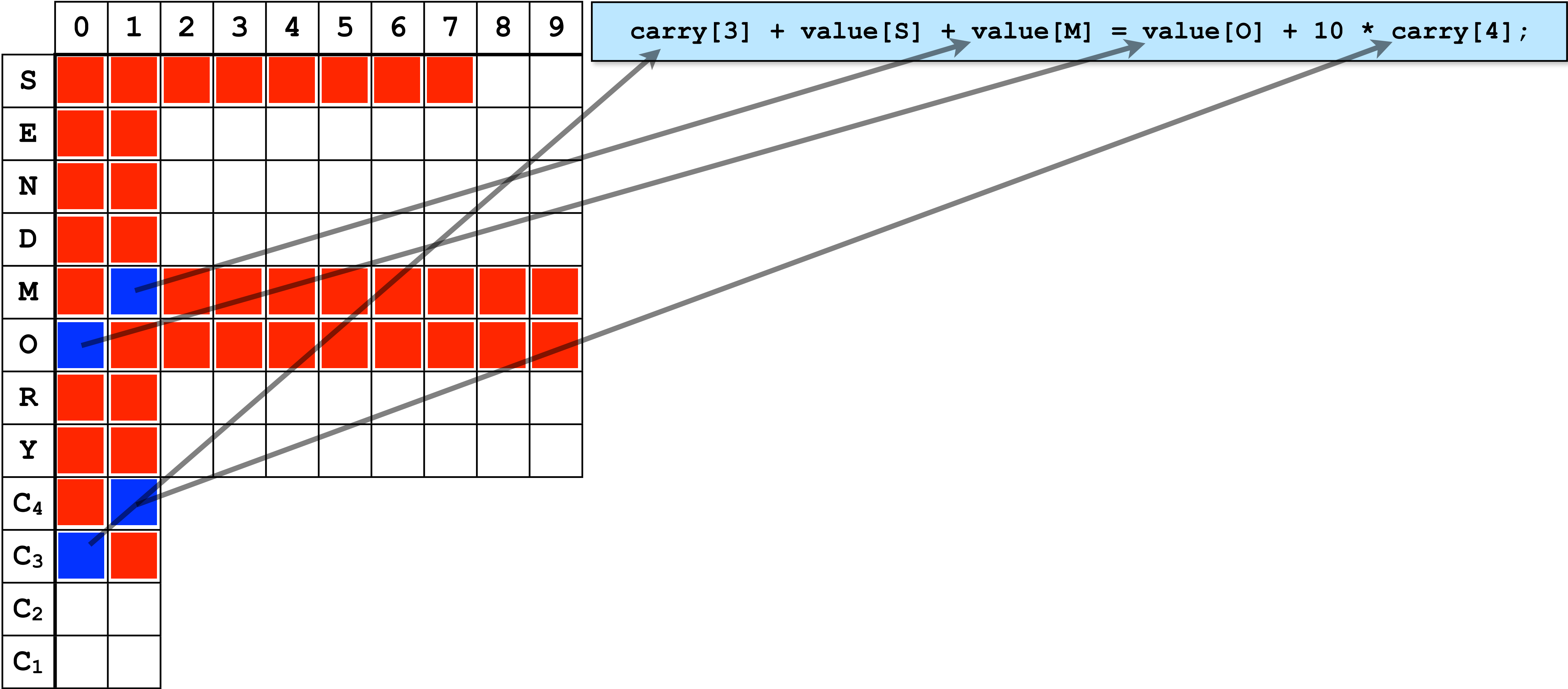
solve {
  forall(i in Letters, j in Letters: i < j)
    value[i] ≠ value[j];
  value[S] ≠ 0;
  value[M] ≠ 0;
  carry[4] = value[M];
  carry[3] ← value[S] + value[M] = value[O] + 10 * carry[4];
  carry[2] + value[E] + value[O] = value[N] + 10 * carry[3];
  carry[1] + value[N] + value[R] = value[E] + 10 * carry[2];
  value[D] + value[E] = value[Y] + 10 * carry[1];
}
```

What is the Search Space?

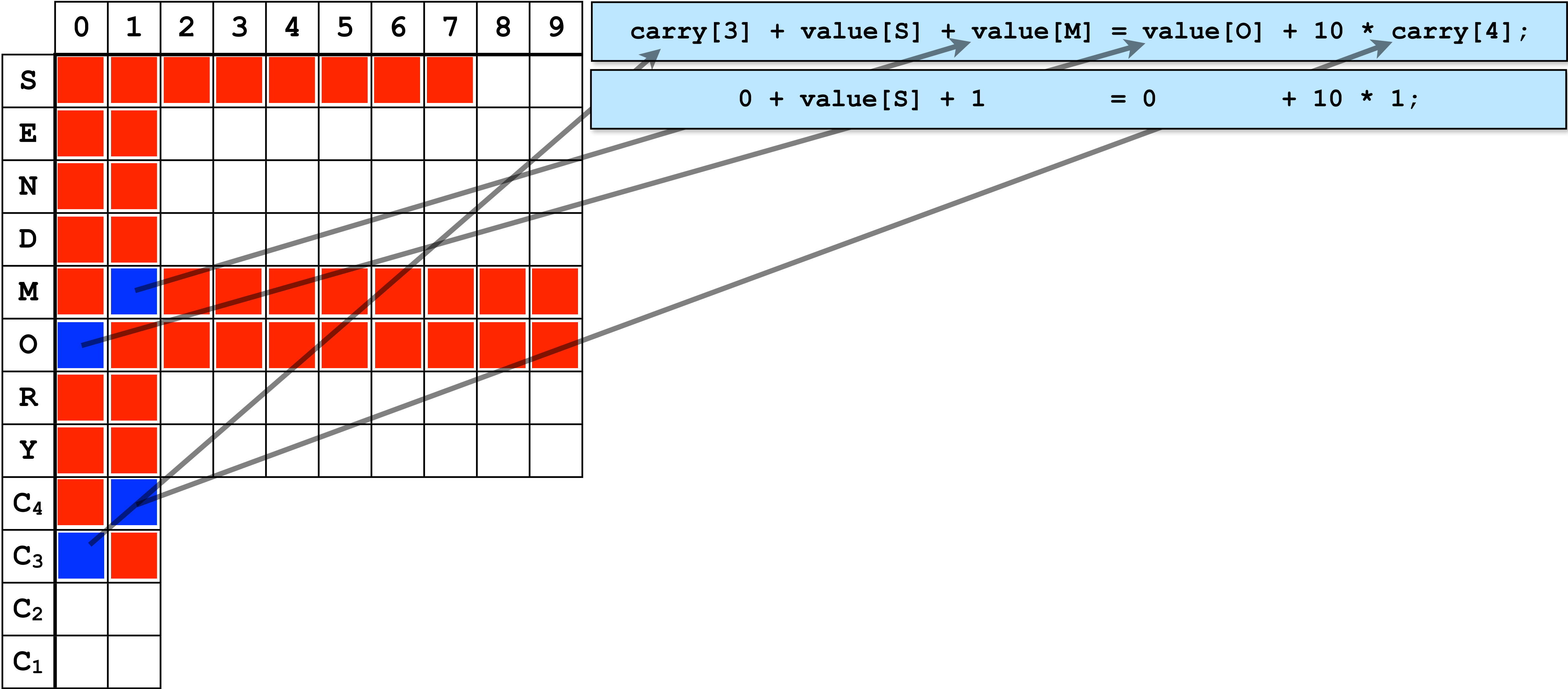
	0	1	2	3	4	5	6	7	8	9
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C ₁										

`carry[3] + value[S] + value[M] = value[0] + 10 * carry[4];`

What is the Search Space?



What is the Search Space?



What is the Search Space?

	0	1	2	3	4	5	6	7	8	9
S										
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$$\text{carry}[3] + \text{value}[\text{S}] + \text{value}[\text{M}] = \text{value}[\text{O}] + 10 * \text{carry}[4];$$

$$0 + \text{value}[\text{S}] + 1 = 0 + 10 * 1;$$

$$\text{value}[\text{S}] = 9;$$

What is the Search Space?

	0	1	2	3	4	5	6	7	8	9
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R										
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$carry[3] + value[S] + value[M] = value[0] + 10 * carry[4];$

$0 + value[S] + 1 = 0 + 10 * 1;$

$value[S] = 9;$

What is the Search Space?

	0	1	2	3	4	5	6	7	8	9
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C ₁										

`carry[3] + value[S] + value[M] = value[0] + 10 * carry[4];`

`0 + value[S] + 1 = 0 + 10 * 1;`

`value[S] = 9;`

```
forall(i in Letters, j in Letters: i < j)
    value[i] ≠ value[j];
```


What is the Search Space?

	0	1	2	3	4	5	6	7	8	9
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```
carry[3] + value[S] + value[M] = value[0] + 10 * carry[4];
```

```
0 + value[S] + 1 = 0 + 10 * 1;
```

```
value[S] = 9;
```

```
forall(i in Letters, j in Letters: i < j)  
  value[i] ≠ value[j];
```

What is the Search Space?

	0	1	2	3	4	5	6	7	8	9
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Linear Constraints over Integers

► Consider a constraint

$$a_1x_1 + \dots + a_nx_n \geq b_1y_1 + \dots + b_my_m$$

$a_i, b_j \geq 0$ are constants

x_i, y_j are variables with domains $D(x_i), D(y_j)$

Linear Constraints over Integers

- Consider a constraint

$$a_1x_1 + \dots + a_nx_n \geq b_1y_1 + \dots + b_my_m$$

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- Feasibility test

Linear Constraints over Integers

- Consider a constraint

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$a_i, b_j \geq 0$ are constants

x_i, y_j are variables with domains $D(x_i), D(y_j)$

- Feasibility test

$$a_1 \max(D(x_1)) + \dots + a_n \max(D(x_n)) \geq b_1 \min(D(y_1)) + \dots + b_m \min(D(y_m))$$

Linear Constraints over Integers

► Consider a constraint

$$a_1x_1 + \dots + a_nx_n \geq b_1y_1 + \dots + b_my_m$$

$a_i, b_j \geq 0$ are constants

x_i, y_j are variables with domains $D(x_i), D(y_j)$

► Feasibility test

$$a_1 \max(D(x_1)) + \dots + a_n \max(D(x_n)) \geq b_1 \min(D(y_1)) + \dots + b_m \min(D(y_m))$$

$$l = a_1 \max(D(x_1)) + \dots + a_n \max(D(x_n))$$

$$r = b_1 \min(D(y_1)) + \dots + b_m \min(D(y_m))$$

Linear Constraints over Integers

► Consider a constraint

$$a_1x_1 + \dots + a_nx_n \geq b_1y_1 + \dots + b_my_m$$

$a_i, b_j \geq 0$ are constants

x_i, y_j are variables with domains $D(x_i), D(y_j)$

► Pruning

$$a_ix_i \geq r - (l - a_i \max(D(x_i)))$$

$$x_i \geq \left\lceil \frac{r - (l - a_i \max(D(x_i)))}{a_i} \right\rceil$$

$$y_j \leq \left\lfloor \frac{l - (r - b_j \min(D(y_j)))}{b_j} \right\rfloor$$

Until Next Time