

# Discrete Optimization

## Greedy Algorithms: Part 2

# Goals of the Lecture

- ▶ Examples of greedy algorithms
  - Set Cover
- ▶ Demonstrate the process of designing optimization algorithms and improving them

# Set Cover

- ▶ You have items,  $1..N$
- ▶ You have a bunch of subsets of those items  $S_i$  in  $\{1,2,\dots,N\}$
- ▶ What is *smallest* number of subsets you need to ***cover*** all the items from  $1..N$ ?

# Set Cover – Example

Items 1..9   Subsets  $S_1, S_2, \dots, S_6$

	1	2	3	4	5	6	7	8	9	
$S_1$	X			X						
$S_2$	X	X	X			X	X		X	
$S_3$		X	X			X	X		X	
$S_4$							X	X	X	
$S_5$	X			X	X	X	X			
$S_6$		X	X					X	X	



# Set Cover – Example

Items 1..9    Subsets  $S_1, S_2, \dots, S_6$

	1	2	3	4	5	6	7	8	9	
$S_1$	X			X						
$S_2$	X	X	X			X	X		X	
$S_3$		X	X			X	X		X	
$S_4$							X	X	X	
$S_5$	X			X	X	X	X			
$S_6$		X	X					X	X	

# Set Cover – Example

Items 1..9   Subsets  $S_1, S_2, \dots, S_6$

	1	2	3	4	5	6	7	8	9	
$S_1$	X			X						
$S_2$	X	X	X			X	X		X	
$S_3$		X	X			X	X		X	
$S_4$							X	X	X	
$S_5$	X			X	X	X	X			
$S_6$		X	X					X	X	

# Set Cover - Example

Items 1..9   Subsets  $S_1, S_2, \dots, S_6$

	1	2	3	4	5	6	7	8	9	
$S_1$	X			X						
$S_2$	X	X	X			X	X		X	
$S_3$		X	X			X	X		X	
$S_4$							X	X	X	
$S_5$	X			X	X	X	X			
$S_6$		X	X					X	X	



# Set Cover – Example

Items 1..9    Subsets  $S_1, S_2, \dots, S_6$

	1	2	3	4	5	6	7	8	9	
$S_1$	X			X						
$S_2$	X	X	X			X	X		X	
$S_3$		X	X			X	X		X	
$S_4$							X	X	X	
$S_5$	X			X	X	X	X			
$S_6$		X	X					X	X	



# Set Cover – First Idea

## ► Idea 1:

- trivial feasible solution

- take sets by input order, until all of the items are covered

- gives a upper bound on the objective (a.k.a. a starting point to improve upon)

# Set Cover – First Idea in Action

	1	2	3	4	5	6	7	8	9	
S <sub>1</sub>	X			X						
S <sub>2</sub>	X	X	X			X	X		X	
S <sub>3</sub>		X	X			X	X		X	
S <sub>4</sub>							X	X	X	
S <sub>5</sub>	X			X	X	X	X			
S <sub>6</sub>		X	X					X	X	

# Set Cover – First Idea in Action

	1	2	3	4	5	6	7	8	9	
S <sub>1</sub>	X			X						
S <sub>2</sub>	X	X	X			X	X		X	
S <sub>3</sub>		X	X			X	X		X	
S <sub>4</sub>							X	X	X	
S <sub>5</sub>	X			X	X	X	X			
S <sub>6</sub>		X	X					X	X	



# Set Cover – First Idea in Action

	1	2	3	4	5	6	7	8	9	
S <sub>1</sub>	X			X						
S <sub>2</sub>	X	X	X			X	X		X	
S <sub>3</sub>		X	X			X	X		X	
S <sub>4</sub>							X	X	X	
S <sub>5</sub>	X			X	X	X	X			
S <sub>6</sub>		X	X					X	X	
	X			X						



# Set Cover – First Idea in Action

	1	2	3	4	5	6	7	8	9	
S <sub>1</sub>	X			X						
S <sub>2</sub>	X	X	X			X	X		X	
S <sub>3</sub>		X	X			X	X		X	
S <sub>4</sub>							X	X	X	
S <sub>5</sub>	X			X	X	X	X			
S <sub>6</sub>		X	X					X	X	
	X	X	X	X		X	X		X	

# Set Cover – First Idea in Action

	1	2	3	4	5	6	7	8	9	
S <sub>1</sub>	X			X						
S <sub>2</sub>	X	X	X			X	X		X	
S <sub>3</sub>		X	X			X	X		X	
S <sub>4</sub>							X	X	X	
S <sub>5</sub>	X			X	X	X	X			
S <sub>6</sub>		X	X					X	X	
	X	X	X	X		X	X		X	

# Set Cover – First Idea in Action

	1	2	3	4	5	6	7	8	9	
S <sub>1</sub>	X			X						
S <sub>2</sub>	X	X	X			X	X		X	
S <sub>3</sub>		X	X			X	X		X	
S <sub>4</sub>							X	X	X	
S <sub>5</sub>	X			X	X	X	X			
S <sub>6</sub>		X	X					X	X	
	X	X	X	X		X	X	X	X	



# Set Cover – First Idea in Action

	1	2	3	4	5	6	7	8	9	
S <sub>1</sub>	X			X						
S <sub>2</sub>	X	X	X			X	X		X	
S <sub>3</sub>		X	X			X	X		X	
S <sub>4</sub>							X	X	X	
S <sub>5</sub>	X			X	X	X	X			
S <sub>6</sub>		X	X					X	X	
	X	X	X	X	X	X	X	X	X	



# Set Cover – First Idea in Action

5	1	2	3	4	5	6	7	8	9
S <sub>1</sub>	X			X					
S <sub>2</sub>	X	X	X			X	X		X
S <sub>3</sub>		X	X			X	X		X
S <sub>4</sub>							X	X	X
S <sub>5</sub>	X			X	X	X	X		
S <sub>6</sub>		X	X					X	X
	X	X	X	X	X	X	X	X	X

# Set Cover – Take Two

- ▶ Idea 1: Objective 5
  - It's better than 6 at least...

# Set Cover – Take Two

- ▶ Idea 1: Objective 5
  - It's better than 6 at least...
- ▶ Idea 2:
  - Take the sets with the most elements first!
  - Intuition, lots of elements will cover quicker

# Set Cover – Second Idea in Action

	1	2	3	4	5	6	7	8	9	
S <sub>1</sub>	X			X						
S <sub>2</sub>	X	X	X			X	X		X	
S <sub>3</sub>		X	X			X	X		X	
S <sub>4</sub>							X	X	X	
S <sub>5</sub>	X			X	X	X	X			
S <sub>6</sub>		X	X					X	X	



# Set Cover – Second Idea in Action

	1	2	3	4	5	6	7	8	9	
S <sub>1</sub>	X			X						2
S <sub>2</sub>	X	X	X			X	X		X	6
S <sub>3</sub>		X	X			X	X		X	5
S <sub>4</sub>							X	X	X	3
S <sub>5</sub>	X			X	X	X	X			5
S <sub>6</sub>		X	X					X	X	4

# Set Cover – Second Idea in Action

	1	2	3	4	5	6	7	8	9	
S <sub>1</sub>	X			X						2
S <sub>2</sub>	X	X	X			X	X		X	6
S <sub>3</sub>		X	X			X	X		X	5
S <sub>4</sub>							X	X	X	3
S <sub>5</sub>	X			X	X	X	X			5
S <sub>6</sub>		X	X					X	X	4
	X	X	X			X	X		X	

# Set Cover – Second Idea in Action

	1	2	3	4	5	6	7	8	9	
S <sub>1</sub>	X			X						2
S <sub>2</sub>	X	X	X			X	X		X	6
S <sub>3</sub>		X	X			X	X		X	5
S <sub>4</sub>							X	X	X	3
S <sub>5</sub>	X			X	X	X	X			5
S <sub>6</sub>		X	X					X	X	4
	X	X	X			X	X		X	



# Set Cover – Second Idea in Action

	1	2	3	4	5	6	7	8	9	
S <sub>1</sub>	X			X						2
S <sub>2</sub>	X	X	X			X	X		X	6
S <sub>3</sub>		X	X			X	X		X	5
S <sub>4</sub>							X	X	X	3
S <sub>5</sub>	X			X	X	X	X			5
S <sub>6</sub>		X	X					X	X	4
	X	X	X	X	X	X	X		X	



# Set Cover – Second Idea in Action

	1	2	3	4	5	6	7	8	9	
S <sub>1</sub>	X			X						2
S <sub>2</sub>	X	X	X			X	X		X	6
S <sub>3</sub>		X	X			X	X		X	5
S <sub>4</sub>							X	X	X	3
S <sub>5</sub>	X			X	X	X	X			5
S <sub>6</sub>		X	X					X	X	4
	X	X	X	X	X	X	X	X	X	

# Set Cover – Second Idea in Action

4	1	2	3	4	5	6	7	8	9	
S <sub>1</sub>	X			X						2
S <sub>2</sub>	X	X	X			X	X		X	6
S <sub>3</sub>		X	X			X	X		X	5
S <sub>4</sub>							X	X	X	3
S <sub>5</sub>	X			X	X	X	X			5
S <sub>6</sub>		X	X					X	X	4
	X	X	X	X	X	X	X	X	X	

# Set Cover – Take Three

- ▶ Idea 1: Objective 5
  - It's better than 6 at least...
- ▶ Idea 2: Objective 4
  - It's better than idea 1 at least...
  - can we improve it? make it smarter?

# Set Cover – Take Three

- ▶ Idea 1: Objective 5
  - It's better than 6 at least...
- ▶ Idea 2: Objective 4
  - It's better than idea 1 at least...
  - can we improve it? make it smarter?
- ▶ Idea 3:
  - As we pickup sets, update set priority with just the ***un-covered*** items!



# Set Cover – Third Idea in Action

	1	2	3	4	5	6	7	8	9	
S <sub>1</sub>	X			X						2
S <sub>2</sub>	X	X	X			X	X		X	6
S <sub>3</sub>		X	X			X	X		X	5
S <sub>4</sub>							X	X	X	3
S <sub>5</sub>	X			X	X	X	X			5
S <sub>6</sub>		X	X					X	X	4

# Set Cover – Third Idea in Action

	1	2	3	4	5	6	7	8	9	
S <sub>1</sub>	X			X						2
S <sub>2</sub>	X	X	X			X	X		X	6
S <sub>3</sub>		X	X			X	X		X	5
S <sub>4</sub>							X	X	X	3
S <sub>5</sub>	X			X	X	X	X			5
S <sub>6</sub>		X	X					X	X	4
	X	X	X			X	X		X	

# Set Cover

	1	2	3	4	5	6	7	8	9	
S <sub>1</sub>	X			X						1
S <sub>2</sub>	X	X	X			X	X		X	0
S <sub>3</sub>		X	X			X	X		X	0
S <sub>4</sub>							X	X	X	1
S <sub>5</sub>	X			X	X	X	X			2
S <sub>6</sub>		X	X					X	X	1
	X	X	X			X	X		X	



# Set Cover

	1	2	3	4	5	6	7	8	9	
S <sub>1</sub>	X			X						1
S <sub>2</sub>	X	X	X			X	X		X	0
S <sub>3</sub>		X	X			X	X		X	0
S <sub>4</sub>							X	X	X	1
S <sub>5</sub>	X			X	X	X	X			2
S <sub>6</sub>		X	X					X	X	1
	X	X	X	X	X	X	X		X	

# Set Cover

	1	2	3	4	5	6	7	8	9	
S <sub>1</sub>	X			X						0
S <sub>2</sub>	X	X	X			X	X		X	0
S <sub>3</sub>		X	X			X	X		X	0
S <sub>4</sub>							X	X	X	1
S <sub>5</sub>	X			X	X	X	X			0
S <sub>6</sub>		X	X					X	X	1
	X	X	X	X	X	X	X		X	

# Set Cover

	1	2	3	4	5	6	7	8	9	
S <sub>1</sub>	X			X						0
S <sub>2</sub>	X	X	X			X	X		X	0
S <sub>3</sub>		X	X			X	X		X	0
S <sub>4</sub>							X	X	X	1
S <sub>5</sub>	X			X	X	X	X			0
S <sub>6</sub>		X	X					X	X	1
	X	X	X	X	X	X	X	X	X	



# Set Cover

3	1	2	3	4	5	6	7	8	9	
S <sub>1</sub>	X			X						0
S <sub>2</sub>	X	X	X			X	X		X	0
S <sub>3</sub>		X	X			X	X		X	0
S <sub>4</sub>							X	X	X	1
S <sub>5</sub>	X			X	X	X	X			0
S <sub>6</sub>		X	X					X	X	1
	X	X	X	X	X	X	X	X	X	

# Set Cover – Done?

- ▶ Idea 1: Objective 5
  - It's better than 6 at least...
- ▶ Idea 2: Objective 4
  - It's better than idea 1 at least...
  - can we improve it? make it smarter?
- ▶ Idea 3: Objective 3
  - It's better than idea 2! More progress.
  - What is the price?

# Set Cover – Done?

- ▶ Idea 1: Objective 5
  - It's better than 6 at least...
- ▶ Idea 2: Objective 4
  - It's better than idea 1 at least...
  - can we improve it? make it smarter?
- ▶ Idea 3: Objective 3
  - It's better than idea 2! More progress.
  - What is the price?
- ▶ Can we do better? Is 3 optimal?



# Set Cover – 2 is Optimal

2	1	2	3	4	5	6	7	8	9	
S <sub>1</sub>	X			X						
S <sub>2</sub>	X	X	X			X	X		X	
S <sub>3</sub>		X	X			X	X		X	
S <sub>4</sub>							X	X	X	
S <sub>5</sub>	X			X	X	X	X			
S <sub>6</sub>		X	X					X	X	
	X	X	X	X	X	X	X	X	X	

# Set Cover – Getting Stuck

- ▶ Out of clever ideas
- ▶ How can we do better?
- ▶ Look at the example in more detail

# Set Cover – Insights

	1	2	3	4	5	6	7	8	9	
S <sub>1</sub>	X			X						
S <sub>2</sub>	X	X	X			X	X		X	
S <sub>3</sub>		X	X			X	X		X	
S <sub>4</sub>							X	X	X	
S <sub>5</sub>	X			X	X	X	X			
S <sub>6</sub>		X	X					X	X	
	1	2	3	4	5	6	7	8	9	



# Set Cover – Insights

	1	2	3	4	5	6	7	8	9	
S <sub>1</sub>	X			X						
S <sub>2</sub>	X	X	X			X	X		X	
S <sub>3</sub>		X	X			X	X		X	
S <sub>4</sub>							X	X	X	
S <sub>5</sub>	X			X	X	X	X			
S <sub>6</sub>		X	X					X	X	

# Set Cover – Insights

$S_2$  dominates  $S_3$

	1	2	3	4	5	6	7	8	9	
$S_1$	X			X						
$S_2$	X	X	X			X	X		X	
$S_3$		X	X			X	X		X	
$S_4$							X	X	X	
$S_5$	X			X	X	X	X			
$S_6$		X	X					X	X	

# Set Cover – Insights

$S_2$  dominates  $S_3$

	1	2	3	4	5	6	7	8	9	
$S_1$	X			X						
$S_2$	X	X	X			X	X		X	
$S_3$										
$S_4$							X	X	X	
$S_5$	X			X	X	X	X			
$S_6$		X	X					X	X	



# Set Cover – Insights

$S_2$  dominates  $S_3$

	1	2	3	4	5	6	7	8	9	
$S_1$	X			X						
$S_2$	X	X	X			X	X		X	
$S_3$										
$S_4$							X	X	X	
$S_5$	X			X	X	X	X			
$S_6$		X	X					X	X	

# Set Cover – Insights

S<sub>2</sub> dominates S<sub>3</sub>

S<sub>5</sub> is required

	1	2	3	4	5	6	7	8	9	
S <sub>1</sub>	X			X						
S <sub>2</sub>	X	X	X			X	X		X	
S <sub>3</sub>										
S <sub>4</sub>							X	X	X	
S <sub>5</sub>	X			X	X	X	X			
S <sub>6</sub>		X	X					X	X	

# Set Cover – Doing Better

- ▶ Remove  $S_3$
- ▶ Start with  $S_5$  included
- ▶ These kinds of deductions are the essence of Constraint Programming
  - one example of how to go beyond greedy
  - you will learn many ways in this class



Until Next Time