Exploring Quantum Physics

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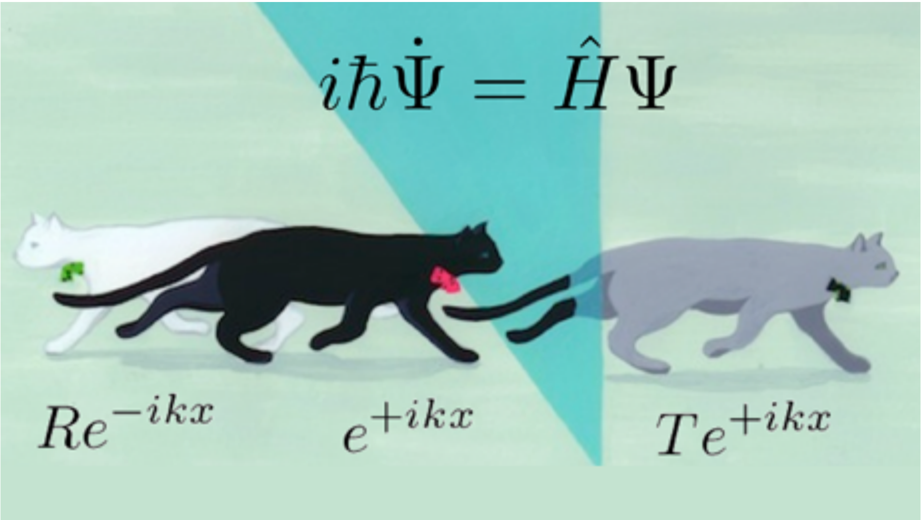
Symmetry in Quantum Physics

Part VIII. Angular Momentum – More Explicit Solutions



Symmetry in quantum physics

$$L_u = \frac{\hbar}{i}\epsilon_{uvw}x_v\partial_w$$

$$[L_u, L_v] = i\hbar\epsilon_{uvw}L_w$$

$$[L_u, x_v] = i\hbar\epsilon_{uvw}x_w$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{L^2}{\hbar^2 r^2}$$

We previously found that any purely radial function $f(r)$ has zero angular momentum.

Let's look among familiar things to find functions with definite, *nontrivial* angular momentum.

How about x_1, x_2 and x_3?

$$L_u x_a = \epsilon_{uvw}x_v p_w x_a = \epsilon_{uvw}x_v\frac{\hbar}{i}\delta_{wa} = \frac{\hbar}{i}\epsilon_{uva}x_v$$

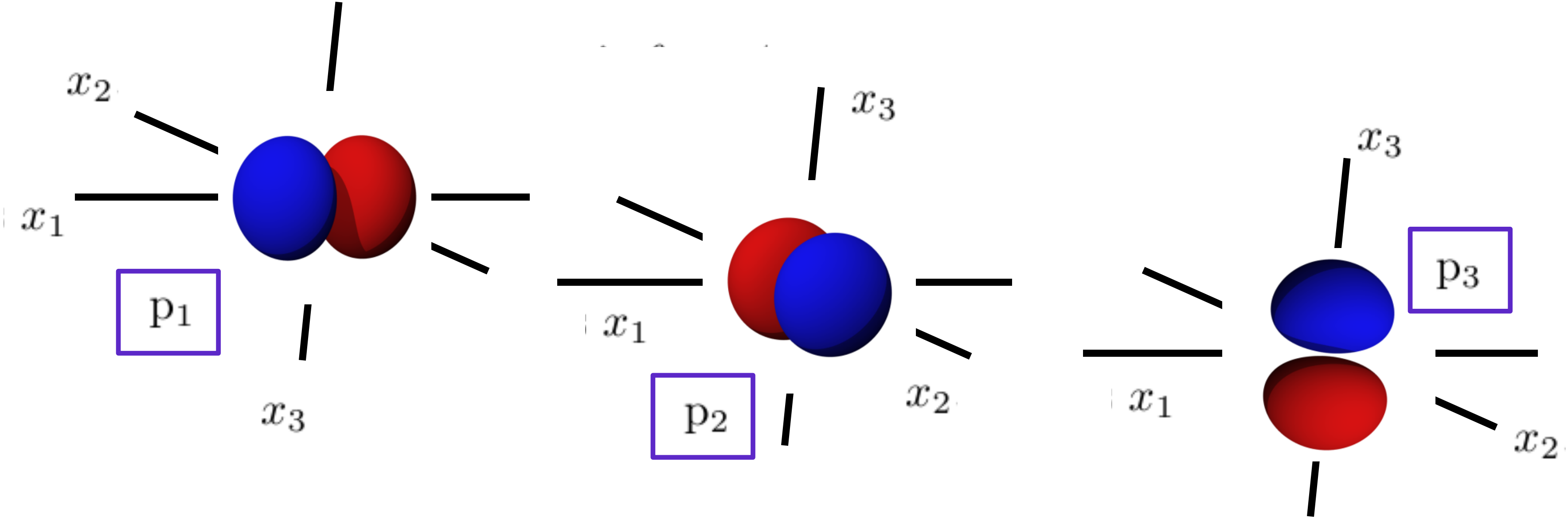
$$L^2 x_a = L_u L_u x_a = \epsilon_{ust}x_s p_t L_u x_a = \epsilon_{ust}x_s p_t\frac{\hbar}{i}\epsilon_{uva}x_v = \epsilon_{ust}\epsilon_{uva}x_s\left(\frac{\hbar}{i}\right)^2\delta_{tv}$$

$$= \epsilon_{usv}\epsilon_{uva}x_s\left(-\hbar^2\right) = \left(\hbar^2\right)\epsilon_{usv}\epsilon_{uav}x_s = 2\hbar^2 x_a = \hbar^2 l(l+1)x_a \text{ with } l = 1$$

Symmetry in quantum physics

x_1, x_2, x_3 are eigenfunctions of L^2 with eigenvalues $l(l+1)\hbar^2$ with $l = 1$

Multiply by $\exp(-r^2/2d^2)$ to get orbitals $\mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_3$



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Invideo quiz

\noindent x_1, x_2, x_3 are three eigenfunctions of L^2 that have the same eigenvalues, $2 \hbar^2 = l(l+1) \hbar^2$ with $l = 1$
\vspace {8 pt}
\noindent Keep in mind the identities $\left[L_u,x_v \right]= i \hbar \epsilon_{uvw} x_w$ and $r^2 = x_1^2+x_2^2+x_3^2$.
\vspace {8 pt}
Which of the following statements are true?
\vspace {8 pt}
The eigenfunction x_k is simultaneously an eigenfunction of the operator L_k, with eigenvalue 0. True
\vspace {8 pt}
The three eigenfunctions x_k have the same parity under inversion of all coordinates. True
\vspace {8 pt}

Symmetry in quantum physics

x_1, x_2, x_3 are eigenfunctions of L^2 with eigenvalues $l(l+1)\hbar^2$ with $l = 1$

Another choice of the three eigenfunctions :

$$\chi_{1,1} = -\tfrac{x_1 + i x_2}{\sqrt{2}}$$

$$\chi_{1,0} = x_3$$

$$\chi_{1,-1} = \tfrac{x_1 - i x_2}{\sqrt{2}}$$

$$L_3 \chi_{1m} = m\hbar \chi_{1,m}$$

$$(L_1 \pm i L_2)\, \chi_{1,m} \equiv L_{\pm} \chi_{1,m} = \sqrt{2 - m\,(m \pm 1)}\chi_{1,m\pm1}$$

$$L_3 \left(\chi_{1,1}\right)^l = l\hbar \left(\chi_{1,1}\right)^l$$

$$L^2 \left(\chi_{1,1}\right)^l = l(l+1)\hbar^2 \left(\chi_{1,1}\right)^l$$

Symmetry in quantum physics

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$$L_3 \left(\chi_{1,1}\right)^l = l\hbar \left(\chi_{1,1}\right)^l$$

$$L^2 \left(\chi_{1,1}\right)^l = l(l+1)\hbar^2 \left(\chi_{1,1}\right)^l$$

Construct general $|l, m>$ as follows :

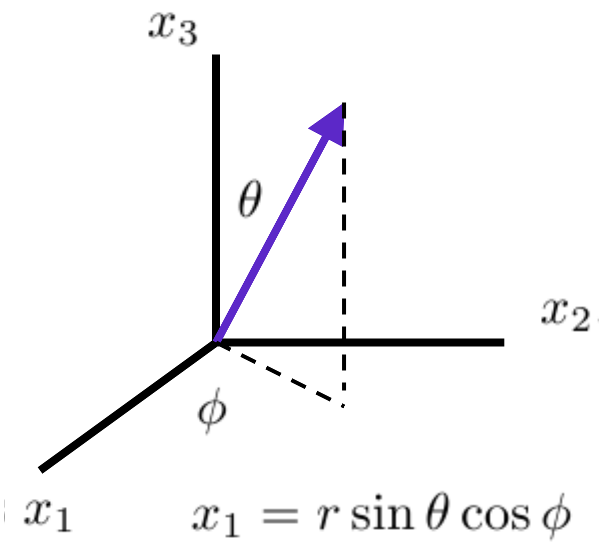
$|l, l > \left(\chi_{1,1}\right)^l$

$L_{-}|l, l >= \sqrt{l(l+1) - m(m-1)}|l, l-1 > \quad m = l$

Repeat application of L_{-}

$$|l, l >= \sum_{a+b+c=l} U_{abc} x_1^a x_2^b x_3^c$$

Symmetry in quantum physics



$$x_1 = r \sin\theta \cos\phi$$

$$x_2 = r \sin\theta \sin\phi$$

$$x_3 = r \cos\theta$$

We constructed angular momentum functions from polynomials in the spatial coordinates. However, all powers of *r* can be removed to get set of dimensionless wavefunctions in angles (θ,ϕ) only. These are the so-called **spherical harmonics** (for which multiple conventions exist):

$$|l, m >= Y_{l,m}\left(\theta, \phi\right)$$

http://dlmf.nist.gov/14.30

Universal for all systems!

invideo quiz
Definitions:
$\vec{L} = \vec{r} \times \vec{p} = \frac{\hbar}{i}
\vec{r} \times \vec{\nabla}= \frac{\hbar}{i}
\epsilon_{uvw}\hat{\mathrm{\bf e}}_u x_v
\frac{\partial}{\partial x_w} \equiv \frac{\hbar}{i}
\epsilon_{uvw}\hat{\mathrm{\bf e}}_u x_v \partial_w
$
\vspace {8 pt}
$L^2 = L_1^2 +L_2^2 + L_3^2 \equiv L_kL_k$
The eigenvalues of L^2 are $\hbar^2 l(l+1)$, where $l = 0, 1, 2, . . .$. By convention we designate these as eigenvalues corresponding to ``angular momentum $l $'', which is the maximum value that can be measured for any component of the angular momentum, in units of $\hbar$. For example, $l = 1$

Symmetry in quantum physics

http://www.fxguide.com