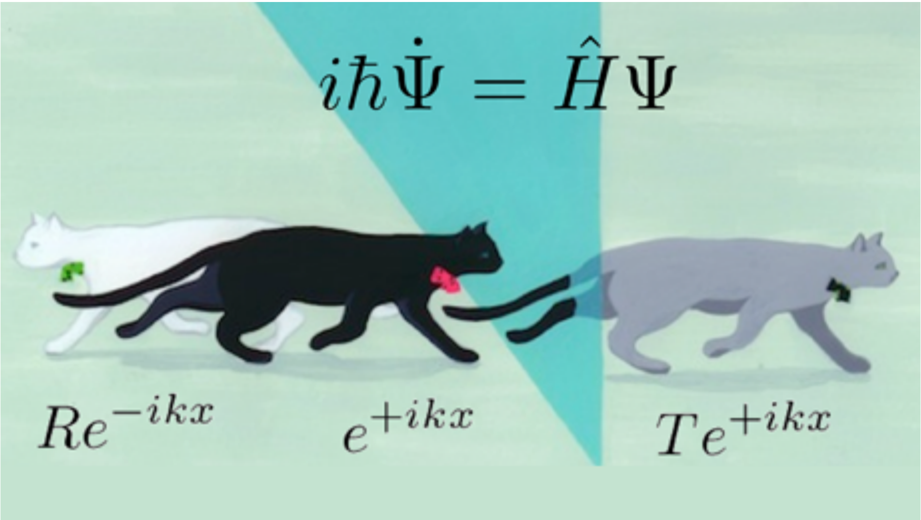
Exploring Quantum Physics

Coursera, Spring 2013 Instructors: Charles W. Clark and Victor Galitski

Symmetry in Quantum Physics

Part VII. Angular Momentum – Explicit Solutions



Symmetry in quantum physics

$$\epsilon_{ijk} = +1 \qquad (ijk) = (123)\,,(231)\,,(312) \qquad \text{cyclic}$$
$$\epsilon_{ijk} = -1 \qquad (ijk) = (213)\,,(132)\,,(321) \qquad \text{odd}$$
$$\epsilon_{ijk} = 0 \qquad \text{otherwise}$$

$$x, y, z \rightarrow x_1, x_2, x_3 \qquad \hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}} \rightarrow \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$$

$$\vec{r} = x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2 + x_3\hat{\mathbf{e}}_3 \equiv x_j\hat{\mathbf{e}}_j$$

$$r = ||\vec{r}|| = \sqrt{x_1^2 + x_2^2 + x_3^2} \equiv \sqrt{x_j x_j}$$

$$\vec{L} = \vec{r} \times \vec{p} = \frac{\hbar}{i}\vec{r} \times \vec{\nabla} = \frac{\hbar}{i}\epsilon_{uvw}\hat{\mathbf{e}}_u x_v \frac{\partial}{\partial x_w} \equiv \frac{\hbar}{i}\epsilon_{uvw}\hat{\mathbf{e}}_u x_v \partial_w$$

$$L^2 = L_1^2 + L_2^2 + L_3^2 \equiv L_k L_k$$

We now use these to generate eigenfunctions of the angular momemtum operator.

These are highly useful in quantum physics and across science and engineering.

Symmetry in quantum physics

$$L_u = \frac{\hbar}{i}\epsilon_{uvw}x_v\partial_w$$

$$[L_u, L_v] = i\hbar\epsilon_{uvw}L_w$$

$$[L_u, x_v] = i\hbar\epsilon_{uvw}x_w$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{L^2}{\hbar^2 r^2}$$

Now we show that $\Psi(r)$ is an eigenfunction of L_u and L^2, for any $\Psi(r)$

$$L_u\Psi(r) = \frac{\hbar}{i}\epsilon_{uvw}x_v\partial_w\Psi(r)$$

$$\partial_w\Psi(r) = \frac{\partial\Psi(r)}{\partial r}\frac{\partial r}{\partial x_w} = \frac{\partial\Psi(r)}{\partial r}\frac{x_w}{r}$$

$$L_u\Psi(r) = \frac{\hbar}{i}\frac{\partial\Psi(r)}{\partial r}\frac{1}{r}\epsilon_{uvw}x_vx_w$$

Invideo quiz

all the definition of the Levi-Civita symbol. Note that the indices i,j,k range independently over th
3.
pace {8 pt}
psilon_{ijk} = +1 \hspace {24 pt} \left(ijk\right) = \left(123\right), \left(231\right), \left(312\right) \hs
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thrm{odd}$
pace {8 pt}
psilon_{ijk} = 0 \hspace {24 pt} \mathrm{otherwise}$
pace {8 pt}
at is the value of $\epsilon_{uvw} x_v x_w$ \hspace {4 pt}? Hint: check your work by explicitly work
ticular case, say $u = 1$.
pace {8 pt}

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$$L_u\Psi(r) = \frac{\hbar}{i}\frac{\partial\Psi(r)}{\partial r}\frac{1}{r}\epsilon_{uvw} x_v x_w = 0$$

$$(L_u)^2\,\Psi(r) = 0 \qquad L_u L_u \Psi(r) = L^2\Psi(r) = 0$$

Thus $\Psi(r)$ is an eigenfunction of L_3 and L^2, with eigenvalue 0. Let's go choose one.

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$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{L^2}{\hbar^2 r^2}$$

We want a function $\Psi(r)$ of *r* alone that is normalized, i.e.

$$\int\int\int dx_1 dx_2 dx_3 \left|\Psi(r)\right|^2 = 1$$

We did this back in "Solving the Schrödinger Equation, Part I. Simple constructive techniques." There we chose a radial Gaussian, which corresponds to the ground state of the isotropic 3D harmonic oscillator:

$$\Psi(x, y, z) = \frac{1}{\pi^{3/4}d^{3/2}}\exp\left[-\frac{x^2+y^2+z^2}{2d^2}\right]$$

Let's try the other relatively obvious choice: $\Psi(r) = \frac{\exp(-r/a_0)}{\sqrt{\pi a_0^3}}$

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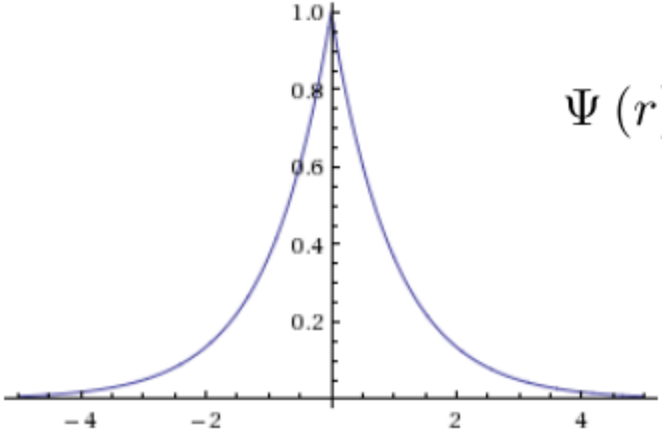

$$\Psi(r) = \frac{\exp(-r/a_0)}{\sqrt{\pi a_0^3}}$$

a section cut along the axis x_1 – note cusp at 0

Use constructive approach to find potential : $-\frac{1}{\Psi}\left[\frac{\hbar^2}{2m_e}\nabla^2\Psi\right] = E - V(r)$

$$V(r) = -\frac{e^2}{r}$$

$$E = -\frac{\hbar^2}{2m_e a_0^2} = -R_\infty hc$$

$$a_0 = \frac{\hbar^2}{m_e e^2}$$

Coulomb potential **1s ground state of hydrogen atom** **Bohr radius**

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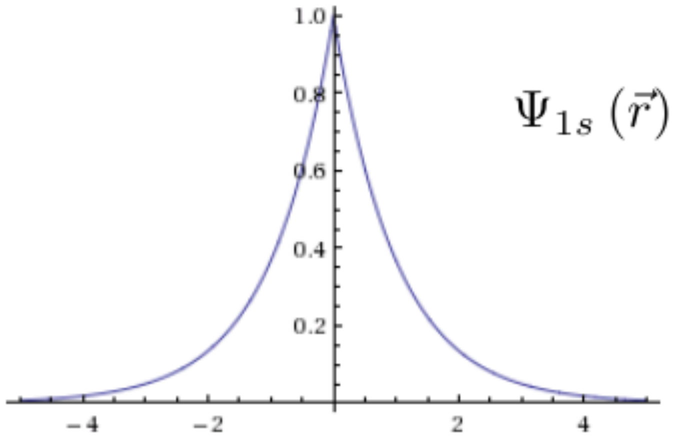

$$\Psi_{1s}(\vec{r}) = \frac{\exp(-r/a_0)}{\sqrt{\pi a_0^3}}$$

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Coulomb potential

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1s ground state of hydrogen atom

$$a_0 = \frac{\hbar^2}{m_e e^2}$$

Bohr radius

Invideo quiz

The Bohr radius, $a_0 = 0.529 177 210 92 (17) \times 10^{-10}$ meter, is the radius of
\vspace {8 pt}
the electron
\vspace {8 pt}
the proton
\vspace {8 pt}
the classical circular orbit of an electron in the field of a proton of infinite mass, which has angular momentum equal to the Planck constant h
\vspace {8 pt}
the classical circular orbit of an electron in the field of a proton of infinite mass, which has angular momentum equal to the reduced Planck constant $\hbar = \frac{h}{2 \pi}$
correct
\vspace {8 pt}

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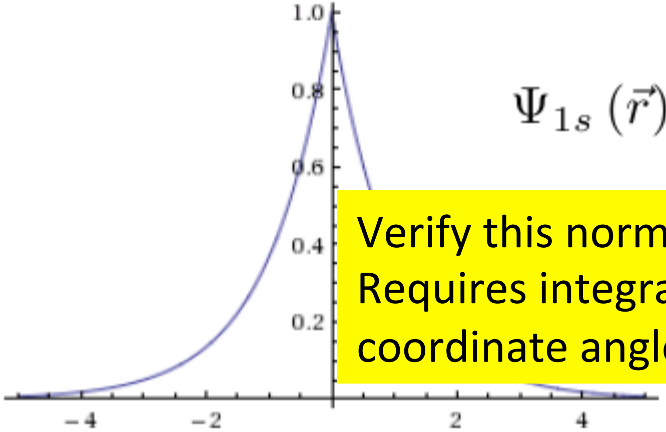

$$\Psi_{1s}(\vec{r}) = \frac{\exp(-r/a_0)}{\sqrt{\pi a_0^3}}$$

Verify this normalization. Requires integration over polar coordinate angles θ,ϕ

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Coulomb potential **1s ground state of hydrogen atom** **Bohr radius**