Joint Quantum Institute

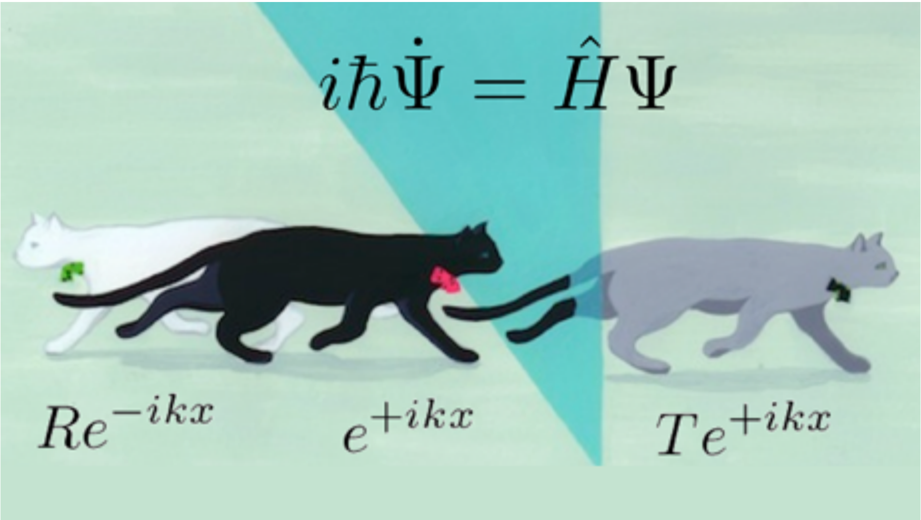
Exploring Quantum Physics

Coursera, Spring 2013 Instructors: Charles W. Clark and Victor Galitski

University of Maryland 1856

Symmetry in Quantum Physics

Part VI. Angular Momentum and the Runge-Lenz Vector



Symmetry in quantum physics

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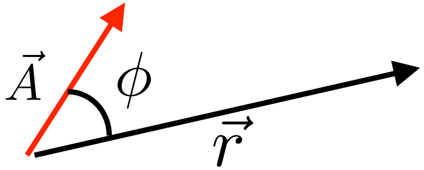
Wolfgang Pauli
1887-1961
1933 Nobel Prize in Physics
"for the discovery of new productive forms of atomic theory".

Say hello to an old friend – the hydrogen atom!

$$E = T + V(r) = \frac{p^2}{2\mu} - \frac{Ze^2}{r} \qquad \vec{L} = \vec{r} \times \vec{p}$$

$$\vec{A} = \vec{p} \times \vec{L} - Ze^2 \mu \hat{r}$$



Pauli derived the symmetric form required for quantum mechanics:

$$\vec{M} = \frac{\vec{p}\times\vec{L}-\vec{L}\times\vec{p}}{2\mu} - Ze^2\frac{\vec{r}}{r}$$

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Commutators of $\vec{L}$ anc $\vec{M}$

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$$\vec{M} = \frac{\vec{p}\times\vec{L}-\vec{L}\times\vec{p}}{2\mu} - Ze^2\frac{\vec{r}}{r}$$

$$H = \frac{p^2}{2\mu} - \frac{Ze^2}{r}$$

$$[L_u, L_v] = i\hbar\epsilon_{uvw}L_w$$

$$[L_u, M_v] = i\hbar\epsilon_{uvw}M_w$$

$$[M_u, M_v] = i\hbar\left(\frac{-2H}{\mu}\right)\epsilon_{uvw}L_w$$

These make it possible to find the bound state energies without solving the Schrödinger equation!

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Symmetry in quantum physics

First, the spectrum associated with $\vec{L}$

Since L_3 and L^2 commute, they have common eigenfunctions. We'll call them $|lm>$

$L_3|lm>= m\hbar|lm>$ From dimensional analysis. Value of *m* to be determined.

$L_+ = L_1 + iL_2$ $\quad L_- = L_1 - iL_2$ $\quad [L_3, L_\pm] = \pm\hbar L_\pm$

$$L_3L_\pm|lm>= L_\pm L_3|lm> \pm\hbar L_\pm|lm>= \hbar(m\pm 1)L_\pm|lm> .$$

"Ladder operators" generate a sequence of eigenstates with equally-spaced eigenvalues. Full analysis shows

$L^2|lm>= \hbar^2 l(l+1)|lm>$ $\quad l = 0, 1, 2...$ $\quad m = -l, -l+1, ...0, ...l-1, l$

Symmetry in quantum physics

Eigenfunctions of angular momentum

The eigenfunctions are functions of the angles only (dimensional analysis)

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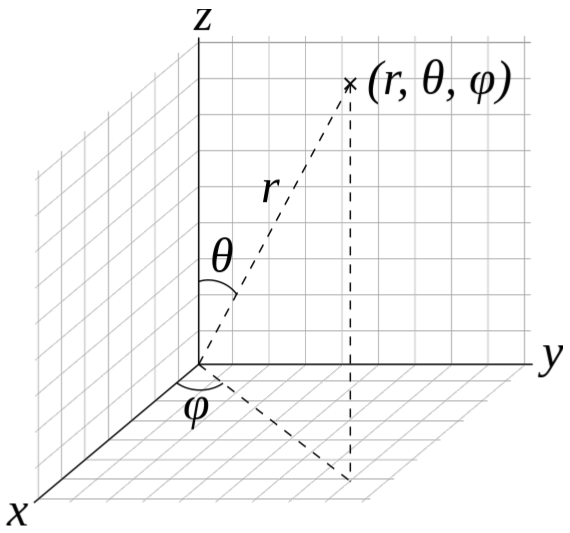
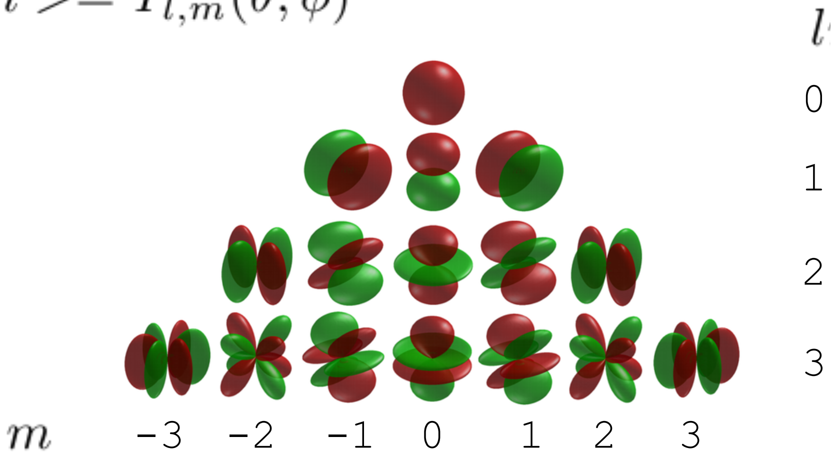


Spherical harmonics (DLMF 14.30)

$$|lm> = Y_{l,m}(\theta, \phi)$$



Symmetry in quantum physics

The *L* and *M* can be combined to form the generators of the SO(4) rotation group in four dimensions (or SO (3,1) for the unbound states.

$$[L_u, L_v] = i\hbar\epsilon_{uvw} L_w$$

$$[L_u, M_v] = i\hbar\epsilon_{uvw} M_w$$

$$[M_u, M_v] = i\hbar\left(\frac{-2H}{\mu}\right)\epsilon_{uvw} L_w$$

A ladder operator approach analogous to that used for the harmonic oscillator gives the spectrum:

$$E(n,l) = -\frac{R_\infty hc(\mu/m_e)}{(n+l+1)^2}$$

$$n = 0, 1, 2, ...$$
$$l = 0, 1, 2, ..., n$$

A total of $(n+l+1)^2$ states of exactly the same energy!

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Quantum Theory: Old and New

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Niels Henrik David Bohr
1885-1962
1922 Nobel Prize in Physics
"for his services in the investigation of the structure of atoms and of the radiation emanating from them".

Radio recombination lines from the largest bound atoms in space

S. V. Stepkin, A. A. Konovalenko, N. G. Kantharia and N. Udaya Shankar, *Monthly Notices of the Royal Astronomical Society* **374**, 852 (2007)

They observe radio-frequency absorption by carbon-like atoms in the cool tenuous medium located in the Perseus arm in front of the supernova remnant, Cassiopeia A (Cas A). These are associated with states with *n* up to 1009: atoms **a million times larger than hydrogen**. The Bohr transition frequencies are now seen to be valid over a range spanning $2.5 \times 10^{15} - 2.6 \times 10^{7}$ Hz: eight decades of frequency!

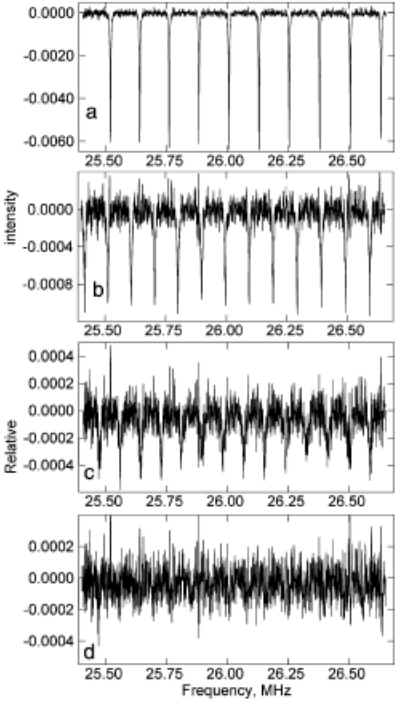

Figure 2. The series of α, β, γ and δ RRLs observed around 26 MHz in the direction of Cas A. Panel (a) shows α series C627. . .C636, panel (b) shows β series C790. . .C802, panel (c) shows γ series C904. . .C917, and panel (d) shows δ series C994. . .C1009.