Exploring Quantum Physics

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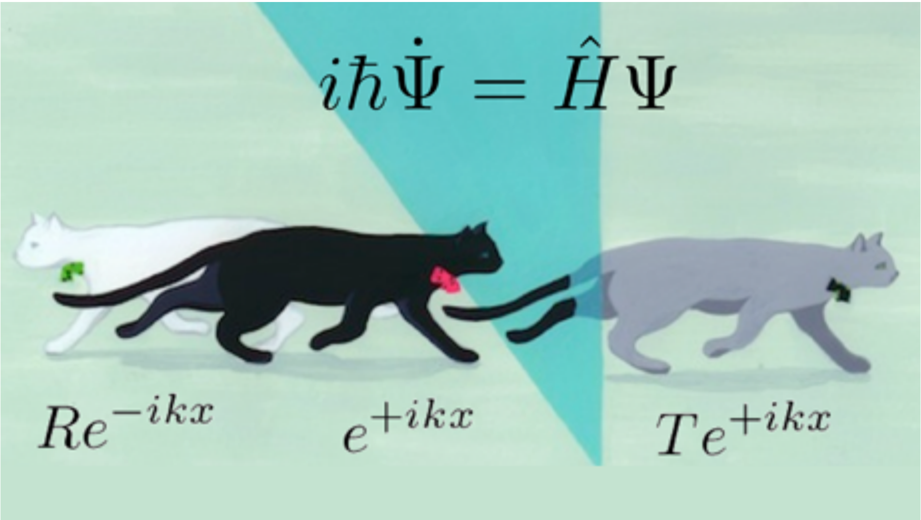
Symmetry in Quantum Physics

Part V. Angular Momentum as an Effective Potential



Symmetry in quantum physics

Eugene Paul Wigner
1902-1995
1963 Nobel Prize in Physics
"for his contributions to the theory of the atomic nucleus and the elementary particles, particularly through the discovery and application of fundamental symmetry principles".

Angular momentum

Now we show how the angular momentum operator contributes to the kinetic energy operator

Again, this is a fundamental and important derivation. You need to work through it yourself. I will show you the steps.

When you have completed this derivation you will have done something foundational that will give you much insight into physics.

Symmetry in quantum physics

$$\vec{L} = \vec{r} \times \vec{p} = \frac{\hbar}{i}\vec{r} \times \vec{\nabla} = \hat{\mathbf{e}}_i L_i = \epsilon_{uvw}\hat{\mathbf{e}}_u x_v p_w$$

$$\boxed{[p_j, x_k] = p_j x_k - x_k p_j = \frac{\hbar}{i}\delta_{jk}}$$

$$[x_a p_j, x_k] = \frac{\hbar}{i}\delta_{jk} x_a$$

$$\boxed{[L_a, x_b] = i\hbar\epsilon_{abc} x_c}$$

$$\boxed{[L_a, p_b] = i\hbar\epsilon_{abc} p_c}$$

$$\boxed{[L_a, L_b] = i\hbar\epsilon_{abc} L_c} \qquad \vec{L} \times \vec{L} = i\hbar\vec{L}$$

$$\boxed{\left[L_i, L^2\right] = 0}$$ where $L^2 = L_i L_i$

The boxed identities all follow directly from the first. You will use them your own way to get the main result of this part.

Symmetry in quantum physics

Now we show the relationship of L^2 to the kinetic energy operator.

$$L^2 = -\hbar^2 \left(\vec{r} \times \vec{\nabla}\right) \cdot \left(\vec{r} \times \vec{\nabla}\right)$$

$$-L^2/\hbar^2 = \epsilon_{ijk} x_j \partial_k \epsilon_{ipq} x_p \partial_q = \epsilon_{ijk} \epsilon_{ipq} x_j \partial_k x_p \partial_q = \left(\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}\right) x_j \partial_k x_p \partial_q$$

$$= x_j \partial_k x_j \partial_k - x_j \partial_k x_k \partial_j$$

further reduce using $\partial_k x_j = \delta_{jk} + x_j \partial_k$

Symmetry in quantum physics

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$$= x_j \partial_k x_j \partial_k - x_j \partial_k x_k \partial_j \qquad = r^2 \nabla^2 - r^2 \frac{\partial^2}{\partial r^2} - 2r \frac{\partial}{\partial r}$$

Symmetry in quantum physics

Thus the Hamiltonian for a particle in a potential $V(x, y, z)$ is

$$\hat{H} = -\frac{\hbar^2}{2M}\nabla^2 + V(x, y, z) = -\frac{\hbar^2}{2M}\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{L^2}{\hbar^2 r^2}\right] + V(x, y, z)$$

Notes:

- We didn't "transform to spherical coordinates". This is how the angular momentum enters the Hamiltonian.
- Anytime you have an equation with a Laplacian, you have something like angular momentum – electrodynamics, acoustics, fluid mechanics, etc.
- Angular momentum is more powerful than gravity! Than electricity! It looks like an inverse-cube vs. inverse square force
- Can generalize to *N* dimensions (hyperspherical coordinates)