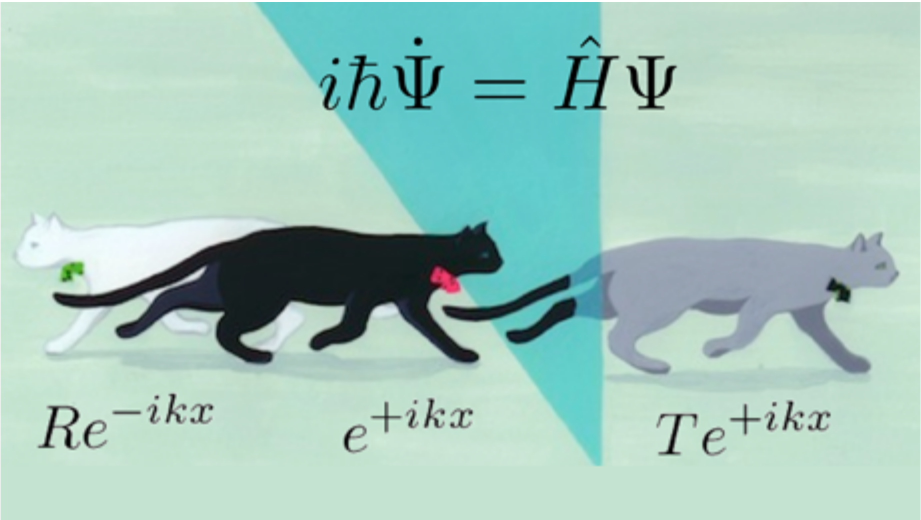
Exploring Quantum Physics

Coursera, Spring 2013 Instructors: Charles W. Clark and Victor Galitski

Symmetry in Quantum Physics

Part III. Angular momentum:

Einstein summation and Levi-Civita symbol



Symmetry in quantum physics

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Eugene Paul Wigner
1902-1995
1963 Nobel Prize in Physics
"for his contributions to the theory of the atomic nucleus and the elementary particles, particularly through the discovery and application of fundamental symmetry principles".

Angular momentum

We are going to derive a number of foundational properties of the angular momentum operator.

We start by reviewing some standard conventions and procedures for tensor calculus.

First used by Albert Einstein in the context of general relativity, these techniques are now applied throughout physics.

Symmetry in quantum physics

We use vector notation but all the symbols denote operators.

$$\vec{L} = \vec{r} \times \vec{p} = \frac{\hbar}{i}\vec{r} \times \vec{\nabla}$$

Three really useful tools, both here and in every other physics subject that involves vector calculus :

1) Notation $\quad x, y, z \to x_1, x_2, x_3 \qquad \hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}} \to \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$

2) Einstein summation convention for repeated indices:

$$\vec{a} \cdot \vec{b} = \sum_{i=1}^{3} a_i b_i$$

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3) Levi-Civita symbol

$$\epsilon_{ijk} = +1 \qquad (ijk) = (123)\,,(231)\,,(312) \qquad \text{cyclic}$$

$$\epsilon_{ijk} = -1 \qquad (ijk) = (213)\,,(132)\,,(321) \qquad \text{odd}$$

$$\epsilon_{ijk} = 0 \qquad \text{otherwise}$$

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Important contraction identity:

$$\epsilon_{ijk}\epsilon_{ist} = \delta_{js}\delta_{kt} - \delta_{jt}\delta_{ks}$$

$$\delta_{js} = 1 \qquad j = s$$

Kronecker delta

$$\delta_{js} = 0 \qquad \text{otherwise}$$

Strongly recommend you pause here and convince yourself of this.

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An example from vector calculus:

$$\boxed{\epsilon_{ijk}\epsilon_{ist} = \delta_{js}\delta_{kt} - \delta_{jt}\delta_{ks}}$$

$$\vec{A} \cdot \left(\vec{B} \times \vec{C}\right)$$

$$A_i \left(B \times C\right)_i$$

$$A_i\epsilon_{ijk}B_jC_k = \epsilon_{ijk}A_iB_jC_k = \epsilon_{jki}B_jC_kA_i = \epsilon_{kij}C_kA_iB_j$$

$$\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) \quad \vec{B} \cdot \left(\vec{C} \times \vec{A}\right) \quad \vec{C} \cdot \left(\vec{A} \times \vec{B}\right)$$

Symmetry in quantum physics

Another example from vector calculus:

$$\boxed{\epsilon_{ijk}\epsilon_{ist} = \delta_{js}\delta_{kt} - \delta_{jt}\delta_{ks}}$$

$$\vec{A} \times \left(\vec{B} \times \vec{C}\right)$$

$$\epsilon_{ijk}\hat{\mathrm{e}}_i A_j \left(B \times C\right)_k$$

$$\epsilon_{ijk}\hat{\mathrm{e}}_i A_j \epsilon_{kst} B_s C_t = \epsilon_{kij}\epsilon_{kst}\hat{\mathrm{e}}_i A_j B_s C_t$$

$$\left(\delta_{is}\delta_{jt} - \delta_{it}\delta_{js}\right)\hat{\mathrm{e}}_i A_j B_s C_t$$

$$\hat{\mathrm{e}}_i B_i A_j C_j - \hat{\mathrm{e}}_i C_i A_j B_j$$

$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \vec{B}\left(\vec{A} \cdot \vec{C}\right) - \vec{C}\left(\vec{A} \cdot \vec{B}\right)$$