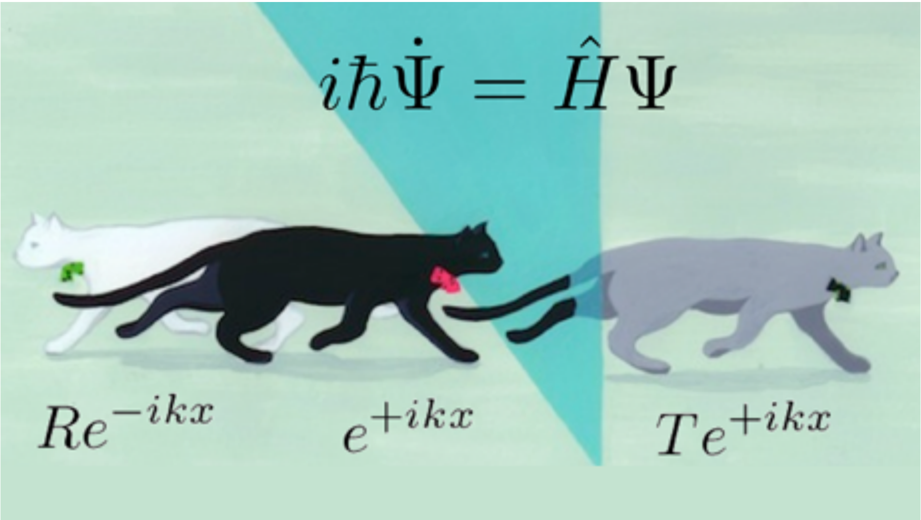
Exploring Quantum Physics

Coursera, Spring 2013 Instructors: Charles W. Clark and Victor Galitski

Quantum harmonic oscillator

Part IV: Harmonic oscillator wave-functions

$\psi(x)$



Summary of what we know so far (a lot)

- Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

- Energy spectrum

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), \text{ with } n = 0, 1, 2, \ldots$$

- The relation between (normalized) eigenfunctions

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle \text{ and } \hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$$

- What we don't know is the explicit form of an eigenfunction...

Deriving the ground-state wave-function

For the ground state, $\underline{\hat{a}^\dagger \hat{a}}\,|0\rangle = 0$, or explicitly

$$\hat{p} = -i\hbar \frac{d}{dx}$$

$$\hat{a}\psi_0(x) \equiv \underline{\left(\sqrt{\frac{m\omega}{2\hbar}}\hat{x} + \frac{i\hat{p}}{\sqrt{2m\omega\hbar}}\right)}\psi_0(x) = 0$$

$$\left(\sqrt{\frac{m\omega}{\cancel{2}\hbar}}\, x + \frac{\sqrt{\hbar}}{\sqrt{\cancel{2}m\omega}} \frac{d}{dx}\right)\psi_0(x) = 0$$

$$\underline{x_0 = \sqrt{\frac{\hbar}{m\omega}}}$$

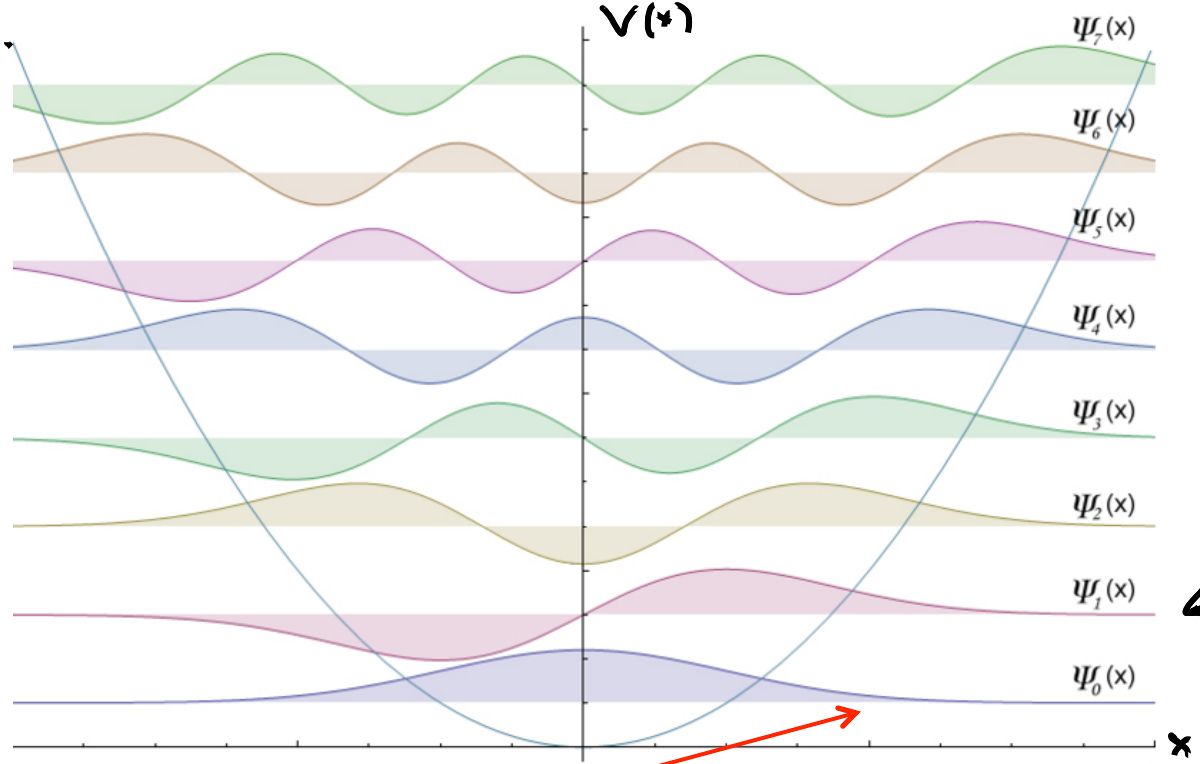
$$\left(\frac{x}{x_0} + x_0 \frac{d}{dx}\right)\psi_0(x) = 0$$

$$\psi_0' = -\frac{x}{x_0^2}\psi_0$$

$$\boxed{\psi_0(x) = C e^{-\frac{x^2}{2x_0^2}}}$$

$$\int_{-\infty}^{+\infty} \psi_0^2(x)\, dx = 1 \implies C = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

Excited states



$$\frac{\hat{a}^+}{\sqrt{n+1}}|n\rangle = |n+1\rangle$$

$$\frac{\hat{a}^+}{\sqrt{n+1}}|0\rangle = |1\rangle$$

Hermite polynomials

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}}$$