Joint Quantum Institute

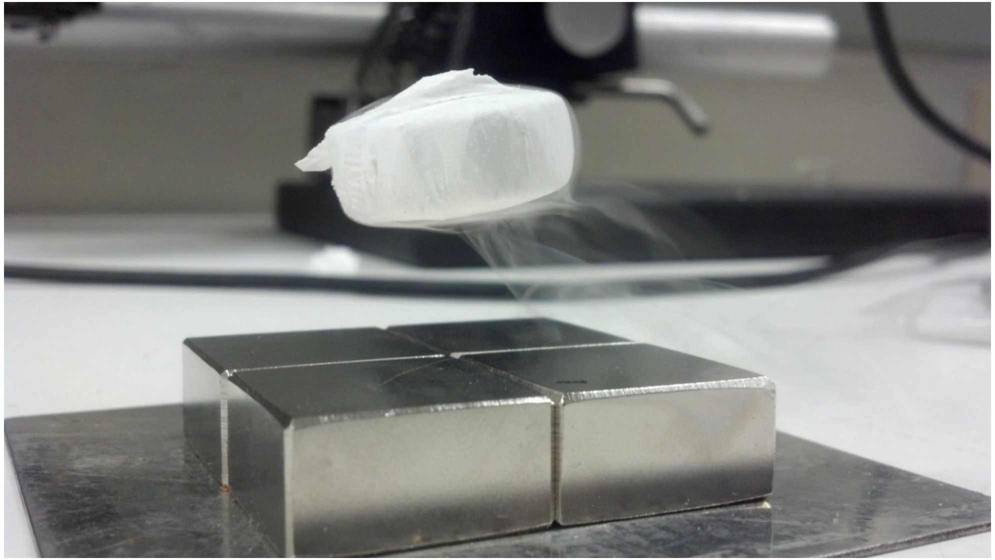
Exploring Quantum Physics

Coursera, Spring 2013 Instructors: Charles W. Clark and Victor Galitski

University of Maryland 1856

Cooper pairing in superconductors

Part I: The phenomenon of superconductivity



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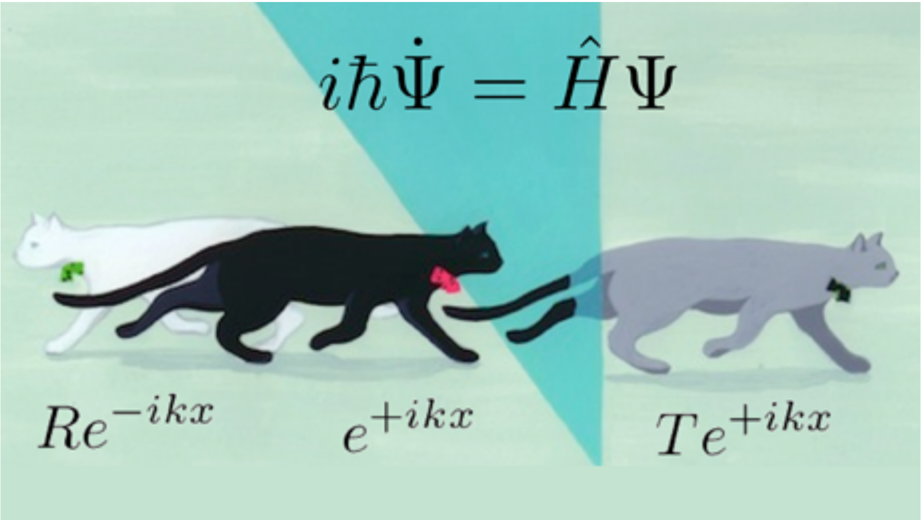
Part III: A step back: two-particle Schrödinger equation



Two-particle Schrödinger equation

- In the next segment, we'll focus on two-electron problem, but keeping in mind other problems later, we now consider a generic case of particles with masses m_1 and m_2, interacting with each other via potential, $V(\vec{r})$:

$\frac{\vec{p}_1^{\,2}}{2m_1}$ $\frac{\vec{p}_2^{\,2}}{2m_2}$ interaction

$$\left[-\frac{\hbar^2\nabla_1^2}{2m_1}-\frac{\hbar^2\nabla_2^2}{2m_2}+V(\vec{r}_1-\vec{r}_2)\right]\psi(\vec{r}_1,\vec{r}_2)=E\psi(\vec{r}_1,\vec{r}_2)$$

- Change of variables: $(\vec{r}_1,\vec{r}_2)\to\left(\vec{R},\vec{r}\right)=\left(\frac{m_1\vec{r}_1+m_2\vec{r}_2}{m_1+m_2},\vec{r}_2-\vec{r}_1\right)$.

- The Schrödinger equation:

$$\left[-\frac{\hbar^2}{2}\left(\frac{1}{m_1}\frac{\partial^2}{\partial\vec{r}_1^{\,2}}+\frac{1}{m_2}\frac{\partial^2}{\partial\vec{r}_2^{\,2}}\right)+V(\vec{r})\right]\tilde{\psi}(\vec{R},\vec{r})=E\tilde{\psi}(\vec{R},\vec{r})$$

Calculating $\nabla_1^2/m_1 + \nabla_2^2/m_2$

- Lets focus on just one (x-) component of the Laplacian with

$$X = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \text{ and } x = x_2 - x_1.$$

- Change of variables in the derivatives:

$$\frac{\partial}{\partial x_1} = \frac{\partial X}{\partial x_1}\frac{\partial}{\partial X} + \frac{\partial x}{\partial x_1}\frac{\partial}{\partial x} = \frac{m_1}{m_1+m_2}\frac{\partial}{\partial X} - \frac{\partial}{\partial x} \text{ and } \frac{\partial}{\partial x_2} = \frac{m_2}{m_1+m_2}\frac{\partial}{\partial X} + \frac{\partial}{\partial x}$$

- Change of variables in the kinetic energy (Laplacians):

$$\frac{1}{m_1}\frac{\partial^2}{\partial x_1^2} + \frac{1}{m_2}\frac{\partial^2}{\partial x_2^2} = \frac{1}{m_1}\left[\frac{m_1}{m_1+m_2}\frac{\partial}{\partial X} - \frac{\partial}{\partial x}\right]^2 + \frac{1}{m_2}\left[\frac{m_2}{m_1+m_2}\frac{\partial}{\partial X} + \frac{\partial}{\partial x}\right]^2$$

$$= \frac{1}{m_1+m_2}\frac{\partial^2}{\partial X^2} + \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\frac{\partial^2}{\partial x^2}$$

$$\frac{m_1 m_2}{m_1 + m_2} = \mu$$

Back to single-particle quantum mechanics

$$-\frac{\hbar^2}{2m_1}\Delta_1 - \frac{\hbar^2}{2m_2}\Delta_2 = -\frac{\hbar^2}{2(m_1+m_2)}\Delta_R - \frac{\hbar^2}{2\mu}\Delta_r$$

$$\mu = \frac{m_1 m_2}{m_1+m_2}$$

$$\left[-\frac{\hbar^2}{2}\left(\underbrace{\frac{\Delta_R}{m_1+m_2}}_{\text{free S. Eq.}} + \underbrace{\frac{\Delta_r}{\mu}\right) + V(\bar r)}_{\bar r}\right]\tilde\Psi(\bar R,\bar r) = E\Psi(\bar R,\bar r)$$

$$\frac{\bar P^2}{2(m_1+m_2)}$$

If $m_1 = m_2 = m$

$m_1+m_2 = 2m$; $\mu = \frac{m^2}{2m} = \frac{m}{2}$

$$\tilde\Psi(\bar R,\bar r) = e^{\frac{i\bar P\cdot\bar R}{\hbar}}\,\underline{\underline{\Psi(\bar r)}}$$

$$\left[-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial \bar r^2} + V(\bar r)\right]\Psi(\bar r) = E'\Psi(\bar r)$$

m