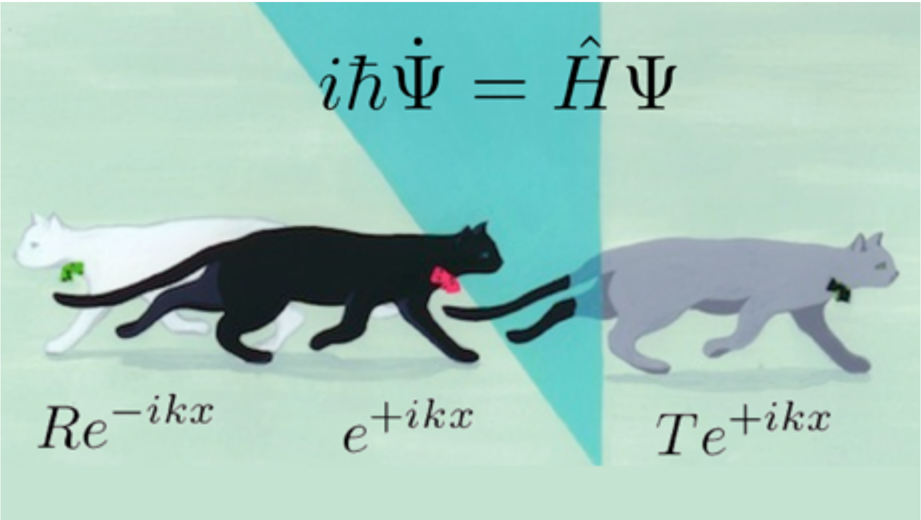
Exploring Quantum Physics

Coursera, Spring 2013 Instructors: Charles W. Clark and Victor Galitski

Bound states in quantum potential wells

Part IV: *Bound states in a delta-potential (cont'd: higher D)*



From 1 to D dimensions

$\vec{r} = (x, y, z, \ldots)$

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \alpha\delta(x)\right]\psi(x) = E\psi(x)$$

F.T.: $\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \ldots$

$$\frac{\hbar^2 k^2}{2m}\tilde{\psi}_k - \alpha\psi(0) = E\tilde{\psi}_k$$

$$\int_k \tilde{\psi}_k = \int_k \frac{\alpha\psi_0}{\frac{\hbar^2 k^2}{2m} + |E|}$$

$$1 = \alpha \int \frac{dk}{2\pi} \frac{1}{\frac{\hbar^2 k^2}{2m} + |E|}$$

$$\left[-\frac{\hbar^2}{2m}\nabla^2 - \alpha\delta(\vec{r})\right]\psi(\vec{r}) = E\psi(\vec{r})$$

F.T.: $\underbrace{\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty}}_{D \text{ times}} \frac{d^D k}{(2\pi)^D} e^{i\vec{k}\cdot\vec{r}} \ldots$

$$\frac{\hbar^2 \vec{k}^2}{2m}\tilde{\psi}_{\vec{k}} - \alpha\psi(0) = E\tilde{\psi}_{\vec{k}}$$

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$$\int_k \tilde{\psi}_k = \int_k \frac{\alpha\psi_0}{\frac{\hbar^2 k^2}{2m} + |E|}$$

$$\frac{1}{\alpha} = \frac{1}{\pi}\int_0^{1/a} dk \, \frac{1}{\frac{\hbar^2 k^2}{2m} + |E|}$$

$$\left[-\frac{\hbar^2}{2m}\nabla^2 - \alpha\delta(\vec{r})\right]\psi(\vec{r}) = E\psi(\vec{r})$$

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$$\frac{\hbar^2 \vec{k}^2}{2m}\tilde{\psi}_{\vec{k}} - \alpha\psi(0) = E\tilde{\psi}_{\vec{k}}$$

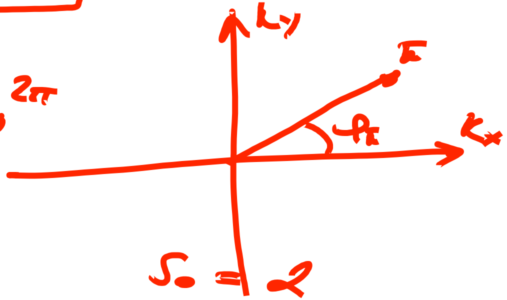
$$\frac{1}{\alpha} = \frac{S_D}{(2\pi)^D}\int_0^{1/a} dk \, \frac{k^{D-1}}{\frac{\hbar^2 k^2}{2m} + |E|}$$

Calculating (and making sense) of the integral

$$1 = \alpha \int \frac{d^D k}{(2\pi)^D} \frac{1}{\frac{\hbar^2 \vec{k}^2}{2m} + |E|}$$

2D: $d^2k = \underline{dk\,k}\; d\varphi_{\vec{k}}$

$$1 = \alpha \int \frac{d^2k}{(2\pi)^2} \dots = \frac{\alpha}{(2\pi)^2} \int_0^{\infty} dk\,k \dots \cancel{\int_0^{2\pi} d\varphi_{\vec{k}}} \; 2\pi$$



$$1 = \frac{\alpha\, S_{D-1}}{(2\pi)^D} \int_0^{1/a} \frac{dk\; k^{D-1}}{\frac{\hbar^2 k^2}{2m} + |E|}$$

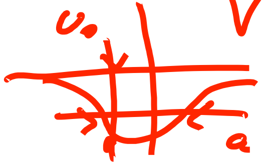
$S_0 = 2$

$S_1 = 2\pi$

$S_2 = 4\pi$

$V(\vec{r}) = \alpha\, \delta(\vec{r})$

$\alpha \sim V_0 a^d$



Critical dimension (=2)

$$\infty \leftarrow \frac{1}{\alpha} = \frac{S_{D-1}}{(2\pi)^D}\int_0^{1/a} dk\, \frac{k^{D-1}}{\frac{\hbar^2k^2}{2m} + |E|}$$

- The main question: is there a bound state in a weak potential ($\alpha \to 0$)?
- Can we make $\int_0^{1/a} dk\, \frac{k^{D-1}}{\frac{\hbar^2k^2}{2m}+|E|}$ arbitrarily large by choosing, E?
- It is equivalent to asking whether the integral below diverges

$$\boxed{\int_0^{1/a} dk\,k^{D-3} = ?}$$

$$D=2 \qquad \int_0^{1/a} \frac{dk}{k} = \ln\frac{1}{0} = \infty$$

- If $D \leq 2$, it diverges or $= \infty$ (there is a bound state), otherwise it is finite (no bound state)

Very shallow level in a 2D quantum well

$$\frac{1}{\alpha} = \frac{1}{2\pi}\int_0^{1/a} dk \frac{k}{\frac{\hbar^2 k^2}{2m} + |E|} = \frac{m}{\pi\hbar^2}\int_0^{1/a} \frac{dk\ k}{k^2 + \left(\frac{\sqrt{2m|E|}}{\hbar}\right)^2} \simeq$$

$$\simeq \frac{m}{\pi\hbar^2}\int_{\frac{\sqrt{2m|E|}}{\hbar}}^{1/a} \frac{dk}{k} = \frac{m}{\pi\hbar^2}\ln\left(\frac{1}{a}\frac{\hbar}{\sqrt{2m|E|}}\right)$$

$$\boxed{|E| \sim \frac{\hbar^2}{ma^2}\exp\left[-\frac{2\pi\hbar^2}{\alpha}\right]_{\alpha\to 0}} \propto e^{-\frac{\#}{\alpha}}$$

$$f(\alpha) \simeq \underset{0}{\underset{\|}{f(0)}} + \underset{0}{\underset{\|}{f'(0)}}\alpha + \underset{0}{\underset{\|}{\frac{f''(0)}{2}}}\alpha^2 + \ldots$$