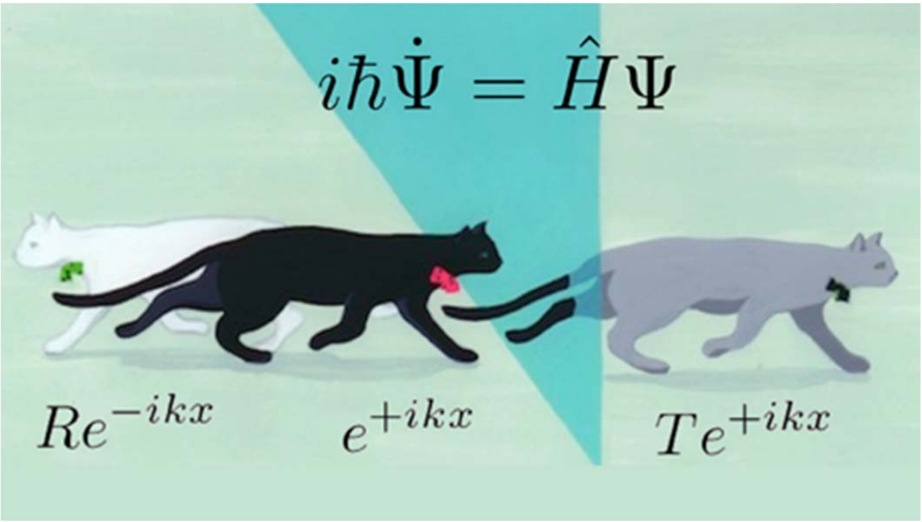
Exploring Quantum Physics

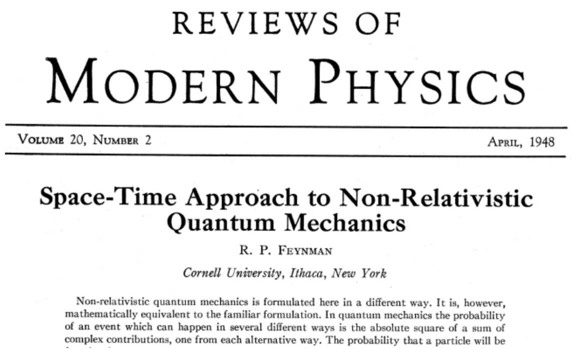
Coursera, Spring 2013 Instructors: Charles W. Clark and Victor Galitski

Using Feynman path integral

Part I: From Feynman to Newton: *classical limit*



From Feynman to Newton



REVIEWS OF

MODERN PHYSICS

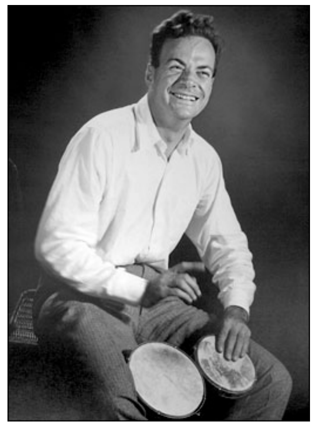
VOLUME 20, NUMBER 2 APRIL, 1948

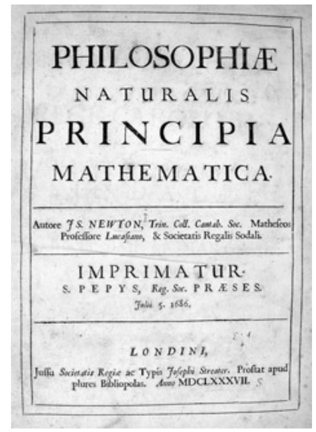
Space-Time Approach to Non-Relativistic Quantum Mechanics

R. P. FEYNMAN

Cornell University, Ithaca, New York

Non-relativistic quantum mechanics is formulated here in a different way. It is, however, mathematically equivalent to the familiar formulation. In quantum mechanics the probability of an event which can happen in several different ways is the absolute square of a sum of complex contributions, one from each alternative way. The probability that a particle will be



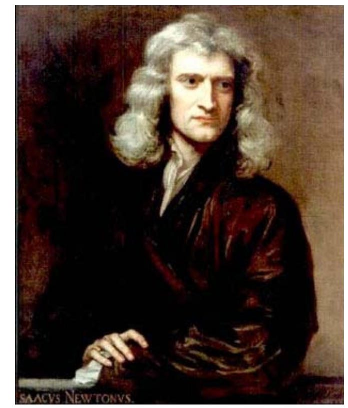


PHILOSOPHIÆ NATURALIS PRINCIPIA MATHEMATICA.

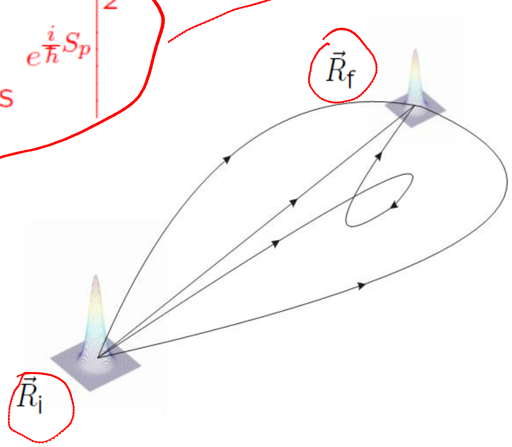
Autore J S. NEWTON, Trin. Coll. Cantab. Soc. Matheseos Professore Lucasiano, & Societatis Regalis Sodali.

IMPRIMATUR. S. PEPYS, Reg. Soc. PRÆSES. Julii 5. 1686.

LONDINI, Jussu Societatis Regiæ ac Typis Josephi Streater. Prostat apud plures Bibliopolas. Anno MDCLXXXVII.



$$w_{i\to f} = \left| \sum_{p \in \text{all paths}} e^{\frac{i}{\hbar} S_p} \right|^2$$



$$m\vec{a} = \vec{F}$$

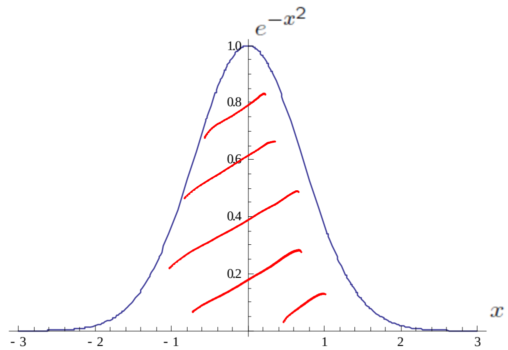
Newton's original version of what we currently call Newton's second law of motion:

"A change in motion is proportional to the motive force impressed and takes place along the straight line in which that force is impressed."

Laplace's method (saddle-point approximation)

- Gaussian integral:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

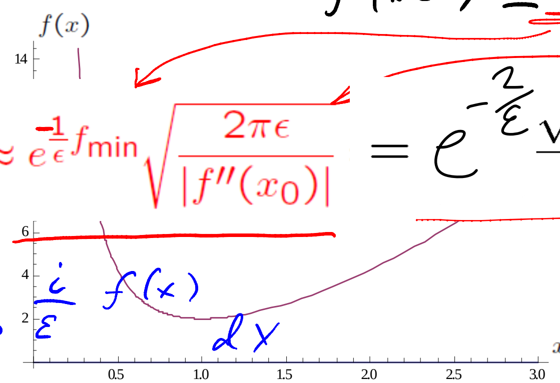


There is no closed analytical expression for $\int e^{-f(x)}dx$. But if there is a small parameter in the exponential, $\int e^{-\frac{1}{\epsilon}f(x)}dx, \epsilon \to 0$, one can simplify things.

$$f(x \sim 1) \simeq f(1) + \underbrace{f'(1)}_{0}(x-1) + \frac{f''(1)}{2}(x-1)^2$$

- Example:

$$I = \int_{-\infty}^{\infty} e^{-\frac{1}{\epsilon}\overbrace{\left[x^2 + 1/x^2\right]}^{f(x)}} dx \approx e^{-\frac{1}{\epsilon} f_{\text{min}}} \sqrt{\frac{2\pi\epsilon}{|f''(x_0)|}} = e^{-\frac{2}{\varepsilon}} \frac{\sqrt{\pi\epsilon}}{2}$$



$$\int e^{\frac{i}{\varepsilon} f(x)} dx$$

The principle of least action

- Quantum effects hinge on interference phenomena. Classical limit implies suppressing them, which happens if

$$\text{wave length} = \lambda = \frac{\hbar}{2\pi p} \to 0 \text{ or, formally, } \hbar \to 0$$

- Feynman path integral in the classical limit

$$\int \mathcal{D}\vec{r}(t)e^{iS/\hbar\to 0}$$

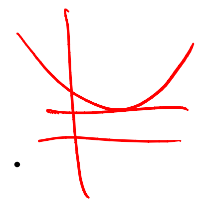
is "collected" from the trajectories for which the action is minimal:

-

$$\hbar = 0 \leftrightarrow \text{ The principle of least action}$$

How to find the special trajectory?

- If we want to determine a point, x_0, where a function, $f(x)$ reaches a minimum, we demand



$$f(x_0 + \delta x) = f(x_0) + \underbrace{f'(x_0)\delta x}_{=0} + \frac{1}{2} f''(x_0)\delta x^2 + \ldots$$

- If we want to determine a trajectory, $x_{\text{cl}}(t)$, on which our functional - action, $S(x)$, - is minimal, we demand:

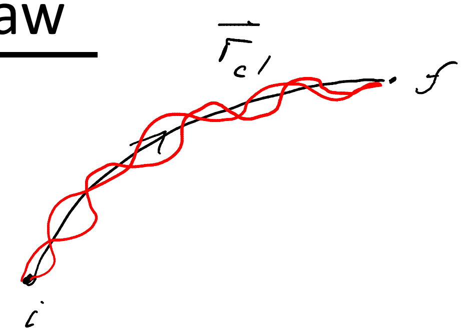
$$S[x_{\text{cl}}(t) + \delta x(t)] = S[x_{\text{cl}}(t)] + \underbrace{\delta S}_{=0} + \ldots$$

- So, we have to set to zero the first variation of the action:

$$\delta S = 0$$

ALL of classical physics is contained in this equation!

Recovering Newton's second law



- Action: $S[\vec{r}(t)] = \int\limits_0^t \left[\frac{m\vec{v}^2}{2} - V(\vec{r})\right] dt$

- Let's calculate its first variation: $\vec v = \frac{d\vec r}{dt} = \dot{\vec r}$

$$S[\vec{r}_{\text{cl}}(t) + \delta\vec{r}(t)] = \int\limits_0^t \left\{\frac{m}{2}\left[\frac{d}{dt}(\vec{r}_{cl} + \delta\vec{r})\right]^2 - V(\vec{r}_{cl} + \delta\vec{r})\right\} dt =$$

$$= \int\limits_0^t \left\{ \underbrace{\frac{m}{2}\dot{\vec r}_{cl}^{\,2} + m\,\dot{\vec r}_{cl}\,\delta\dot{\vec r} - V(\vec r_{cl})}_{S_{cl}} - \frac{\partial V}{\partial \vec r}\cdot\delta\vec r \right\} dt$$

$$= S_{cl} - \int\limits_0^t \underbrace{\left\{ m\ddot{\vec r}_{cl} + \frac{\partial V}{\partial \vec r}\right\}}_{\delta S = 0} \cdot \delta\vec r\, dt$$

$$m\ddot{\vec r}_{cl} = -\frac{\partial V}{\partial \vec r}$$

$$m\vec a = \vec F$$