Calculating the matrix elements

- Reminder (this issue is often a source of confusion): $|\Psi\rangle$ *vs.* $\Psi(x)$
  - $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}px}$
  - $\langle x|x'\rangle = \delta(x - x')$

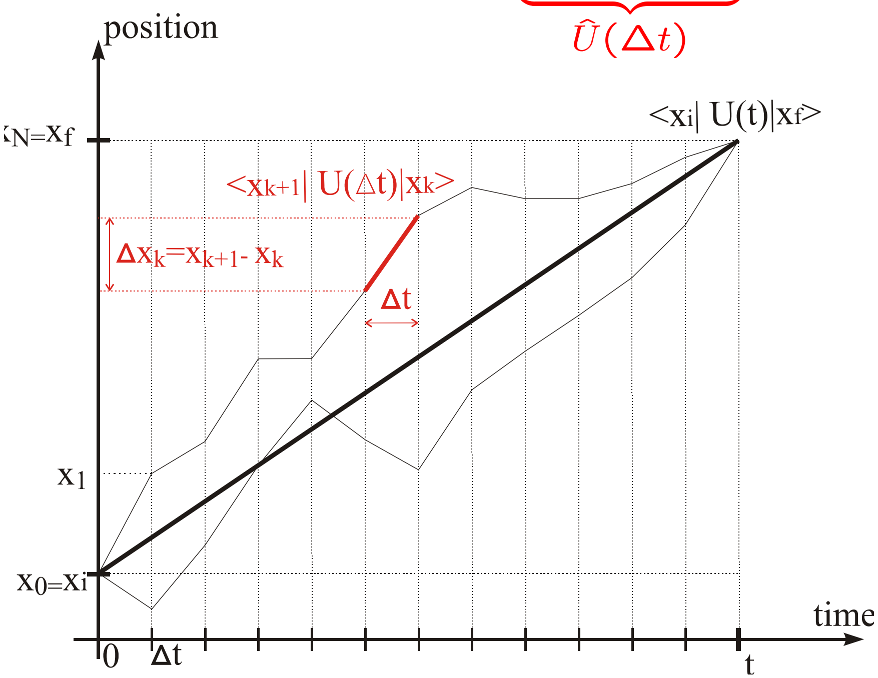
- $\langle x_{k+1}|x_k\rangle = \delta(x_{k+1} - x_k) = \int \frac{dp_k}{2\pi\hbar} e^{i\frac{p_k}{\hbar}(x_{k+1}-x_k)}$

- $\langle x_{k+1}|V(\hat{x})|x_k\rangle = V(x_k)\delta(x_{k+1} - x_k) = V(x_k) \int \frac{dp_k}{2\pi\hbar} e^{i\frac{p_k}{\hbar}(x_{k+1}-x_k)}$

- $\langle x_{k+1}|\frac{\hat{p}^2}{2m}|x_k\rangle = \int dp_k \langle x_{k+1}|\frac{\hat{p}^2}{2m}|p_k\rangle \langle p_k|x_k\rangle = \int \frac{dp_k}{2\pi\hbar} \frac{p_k^2}{2m} e^{i\frac{p_k}{\hbar}(x_{k+1}-x_k)}$

Putting everything together

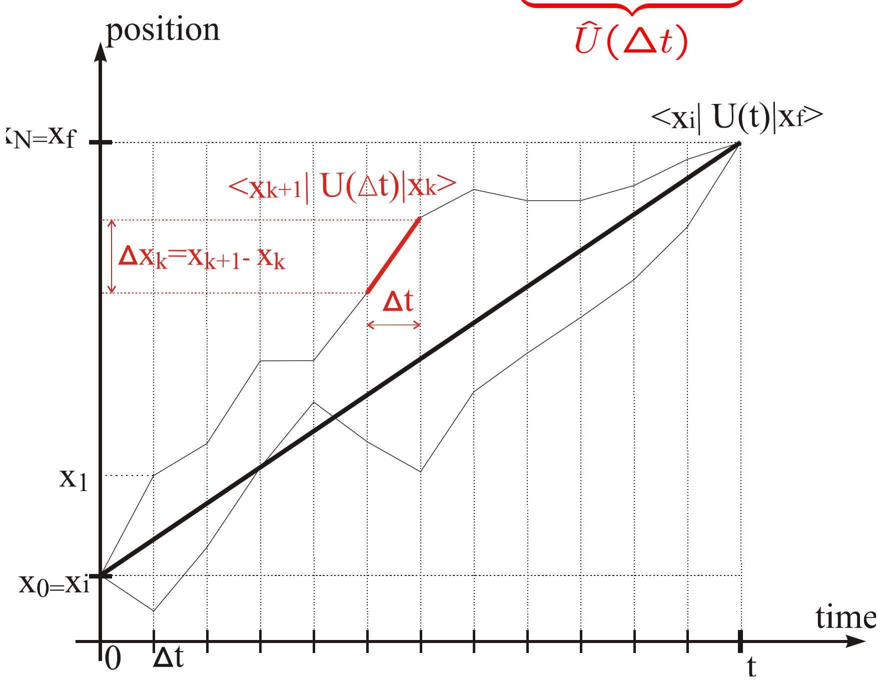
$$\langle x_{k+1}|\underbrace{1-\frac{i}{\hbar}\hat{H}\Delta t}_{\hat{U}(\Delta t)}|x_k\rangle \propto \int dp_k e^{\frac{i}{\hbar}p_k \Delta x_k}\left[1-\frac{i}{\hbar}\left(\frac{p_k^2}{2m}+V(x_k)\right)\Delta t\right]$$



$$\langle x_f|\hat{U}(t)|x_i\rangle = \int \ldots \langle x_2|\hat{U}(\Delta t)|x_1\rangle \, \langle x_1|\hat{U}(\Delta t)|x_i\rangle \, dx_1 dx_2 \ldots$$

Putting everything together

$$\langle x_{k+1}| \underbrace{1-\frac{i}{\hbar}\hat{H}\Delta t}_{\hat{U}(\Delta t)} |x_k\rangle \propto \int dp_k e^{\frac{i}{\hbar}p_k \Delta x_k}\left[1-\frac{i}{\hbar}\left(\frac{p_k^2}{2m}+V(x_k)\right)\Delta t\right]$$



$$\langle x_f|\hat{U}(t)|x_i\rangle = \int dx_1 dx_2 \ldots e^{\frac{i}{\hbar}\mathcal{L}_1 \Delta t} e^{\frac{i}{\hbar}\mathcal{L}_2 \Delta t} \ldots \equiv \int \mathcal{D}x(t) e^{\frac{i}{\hbar}\int_0^t \mathcal{L} dt}$$

Deriving the path integral

- We found

$$\langle x_{k+1}|\, 1-\frac{i}{\hbar}\widehat{H}\Delta t\,|x_k\rangle \propto \int dp_k e^{\frac{i}{\hbar}\left[p_k\frac{\Delta x}{\Delta t}-\frac{p_k^2}{2m}-V(x_k)\right]\Delta t}$$

- As $\Delta t \to 0$, Feynman uses a Taylor expansion "backwards," $1+\epsilon \approx e^{\epsilon}$:

- Putting everything together:

$$G(x_i, x_f; t) \equiv \int \mathcal{D}x(t) e^{\frac{i}{\hbar}\int_0^t \left(\frac{mv^2}{2}-V(x)\right)dt}$$

$$w_{i\to f} = \left|\sum_{p\in \text{all paths}} e^{\frac{i}{\hbar}S_p}\right|^2$$