Exploring Quantum Physics

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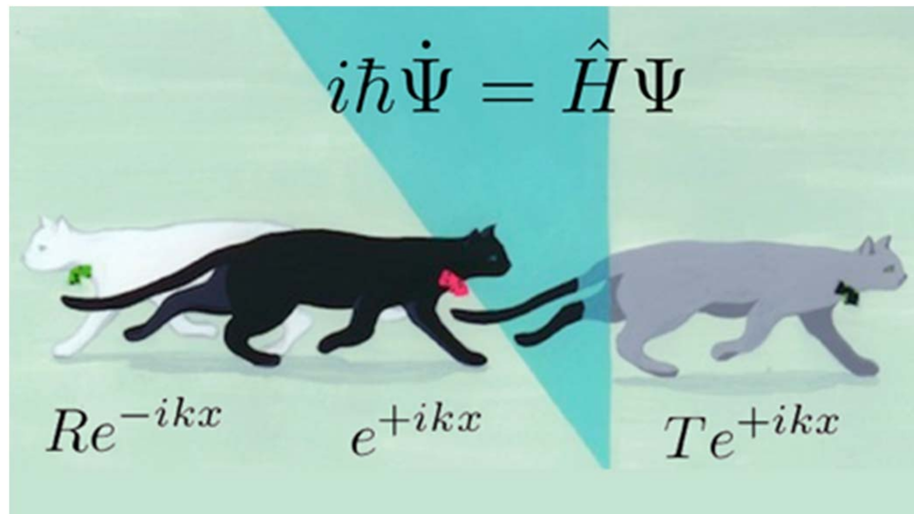
Wave-function and Schrödinger equation

Part IV: Spreading of quantum wave packets



Unstable quantum "particles"

- Assume we have a Gaussian wave-packet at t=0:

$$\Psi(x,0) = Ae^{-\frac{x^2}{2d^2}}$$

- Solving the Schrödinger eq. with this initial condition, shows it's unstable!

$$|\Psi(x,t)|^2 \propto \exp\left[-\frac{x^2}{d^2(1+t^2/\tau^2)}\right], \quad \tau = md^2/\hbar$$

- The size of the wave-packet grows with time, the typical delocalization time

Electron ($m \sim 10^{-27}$ g, $\hbar \sim 10^{-27}$ cm^{2}g/s, $d \sim 10^{-8}$ cm)

$$\tau \sim 10^{-16} \text{ sec}$$

Typical human being ($m \sim 50$ kg, $d \sim 1$ cm):

$$\tau \sim 10^{30} \text{ sec} \sim 10^{23} \text{ years} \sim 10^{13} \text{ lifetimes of the Universe!}$$

Fourier transform and uncertainty relation

- Decompose the wave-packet into plane-waves:

$$\Psi(x,0) = \mathcal{A} e^{-\frac{x^2}{2d^2}} = \int \frac{dp}{(2\pi\hbar)}\, \phi_p\, e^{\frac{i}{\hbar}px}$$

- Wave-function in the momentum space:

$$\phi_p = \int dx \Psi(x,0) e^{-\frac{i}{\hbar}px} \propto e^{-\frac{p^2 d^2}{2\hbar^2}}$$

- Uncertainties:

$$\Delta x \sim d \text{ and } \Delta p \sim \hbar/d$$

Gaussian integral:

$$\int_{-\infty}^{\infty} e^{-\alpha x^2 + \beta x} dx = \sqrt{\frac{\pi}{\alpha}} e^{\frac{\beta^2}{4\alpha}}$$

Manifestation of the Heisenberg uncertainty relation: $\Delta x \cdot \Delta p \gtrsim \hbar$

Spreading out of the wave packet

- Want to solve S.Eq. $i\hbar\partial_t\Psi(x,t) = \frac{\hat{p}^2}{2m}\Psi(x,t)$ with the initial condition:

$$\Psi(x,0) \propto \int dp\, e^{-\frac{p^2 d^2}{2\hbar^2}}\, e^{\frac{i}{\hbar}px}$$

- Observe that if $\Psi_1(x,t)$, $\Psi_2(x,t), \ldots$ are solutions to S.Eq., their sum is also a solution, with the initial condition $\Psi(x,0) = \sum_n \Psi_n(x,0)$.

- Recall that a plane wave, $e^{\frac{i}{\hbar}[px-\varepsilon(p)t]}$, solves S.Eq.

- So, the wave-packet time-evolves as

$$\Psi(x,t) \propto \int dp\, e^{-\frac{p^2 d^2}{2\hbar^2}}\, e^{\frac{i}{\hbar}\left(px-\frac{p^2 t}{2m}\right)} \propto \exp\left[-\frac{x^2}{2d^2}\left(1+\frac{i\hbar t}{md^2}\right)^{-1}\right]$$