Feedback — Problem Set VII

You submitted this homework on **Tue 26 Mar 2013 3:45 PM CDT -0500**. You got a score of **0.00** out of **16.00**. However, you will not get credit for it, since it was submitted past the deadline.

In this problem set, you will be given a total of ten attempts. We will accept late submission until the fifth day after the due date, and late submission will receive half credit. Explanations and answers to the problem set will be available after the due date. Since the homework problems will become gradually more challenging as the course proceeds, we highly recommend you to start the habit of printing out the problems and working on them with paper and pencil. Also, please be sure to read the problem statements carefully and double check your expressions before you submit.

A pdf version of this problem set is available for you to print. Note: all mathematical expressions have to be exact, even when involving constants. Such an expression is required when a function and/or a variable is required in the answer. For example, if the answer is $\sqrt{3}x$, you must type `sqrt(3)*x`, not `1.732*x` for the answer to be graded as being correct.

Question 1

Discrete-Time Fourier Transforms

Find the Fourier transform of the following sequence, where $s(n)$ is some sequence having Fourier transform $S(e^{j2\pi f})$.

$$x(n) = (-1)^n s(n)$$

$X(e^{j2\pi f}) = S(\_\_)$. Provide the expression that fills in the blank for this equation. If $e^{j2\pi f}$ is the answer, type `exp(j*2*pi*f)`.

You entered:

Preview Help

Your Answer	Score	Explanation
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Question Explanation

Using the definition of the DTFT: $X(e^{j2\pi f}) = \sum_n x(n)e^{-j2\pi fn}$, where $x(n) = (-1)^n s(n)$ we get

$$X(e^{j2\pi f}) = \sum_n (-1)^n s(n)e^{-j2\pi fn} = \sum_n s(n)e^{-j2\pi(f-\frac{1}{2})n} = S(e^{j2\pi(f-\frac{1}{2})}).$$

Adding the $(-1)^n$ term in front of $s(n)$ has the effect of rotating the spectrum of $S(e^{j2\pi f})$.

Question 2

Questions 2-5 pertain to the DTFT of the following signal.

$$x(n) = s(n)\cos(2\pi f_0 n)$$

where $s(n)$ is some sequence having Fourier transform $S(e^{j2\pi f})$. The answer is of the form $X(e^{j2\pi f}) = c_1 S(a_1) + c_2 S(a_2)$, where c_1, c_2 are constants and a_1, a_2 are functions of f. For example, if the first term were $S(e^{j2\pi f})$, $c_1 = 1$ and $a_1 = e^{j2\pi f}$. The first term corresponds to the *smallest* frequency component, the second term the largest.

What is c_1? To enter f_0, type `f0`.

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Your Answer	Score	Explanation
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Question Explanation

Using the definition of the DTFT we get

$$X(e^{j2\pi f}) = \sum_n s(n)\cos(2\pi f_0 n)e^{-j2\pi fn} = \sum_n s(n)\frac{1}{2}\left[e^{j2\pi f_0 n} + e^{-j2\pi f_0 n}\right]e^{j2\pi fn}$$
$$= \frac{1}{2}\sum_n s(n)e^{j2\pi(f-f_0)n} + \frac{1}{2}\sum_n s(n)e^{j2\pi(f+f_0)n}$$
$$= \frac{1}{2}S\left(e^{j2\pi(f-f_0)}\right) + \frac{1}{2}S\left(e^{j2\pi(f+f_0)}\right)$$

Question 3

Questions 2-5 pertain to the DTFT of the following signal.

$$x(n) = s(n)\cos(2\pi f_0 n)$$

where $s(n)$ is some sequence having Fourier transform $S(e^{j2\pi f})$. The answer is of the form $X(e^{j2\pi f}) = c_1 S(a_1) + c_2 S(a_2)$, where c_1, c_2 are constants and a_1, a_2 are functions of f. For example, if the first term were $S(e^{j2\pi f})$, $c_1 = 1$ and $a_1 = e^{j2\pi f}$. The first term corresponds to the *smallest* frequency component, the second term the largest.

What is a_1? To enter f_0, type `f0`.

You entered:

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Your Answer	Score	Explanation
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Question Explanation

Using the definition of the DTFT we get

$$X(e^{j2\pi f}) = \sum_n s(n)\cos(2\pi f_0 n)e^{-j2\pi fn} = \sum_n s(n)\frac{1}{2}\left[e^{j2\pi f_0 n} + e^{-j2\pi f_0 n}\right]e^{j2\pi fn}$$
$$= \frac{1}{2}\sum_n s(n)e^{j2\pi(f-f_0)n} + \frac{1}{2}\sum_n s(n)e^{j2\pi(f+f_0)n}$$
$$= \frac{1}{2}S\left(e^{j2\pi(f-f_0)}\right) + \frac{1}{2}S\left(e^{j2\pi(f+f_0)}\right)$$

Question 4

Questions 2-5 pertain to the DTFT of the following signal.

$$x(n) = s(n)\cos(2\pi f_0 n)$$

where $s(n)$ is some sequence having Fourier transform $S(e^{j2\pi f})$. The answer is of the form $X(e^{j2\pi f}) = c_1 S(a_1) + c_2 S(a_2)$, where c_1, c_2 are constants and a_1, a_2 are functions of f. For example, if the first term were $S(e^{j2\pi f})$, $c_1 = 1$ and $a_1 = e^{j2\pi f}$. The first term corresponds to the *smallest* frequency component, the second term the largest.

What is c_2? To enter f_0, type `f0`.

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Question Explanation

Using the definition of the DTFT we get

$$X(e^{j2\pi f}) = \sum_n s(n)\cos(2\pi f_0 n)e^{-j2\pi fn} = \sum_n s(n)\frac{1}{2}\left[e^{j2\pi f_0 n} + e^{-j2\pi f_0 n}\right]e^{j2\pi fn}$$

$$= \frac{1}{2}\sum_n s(n)e^{j2\pi(f-f_0)n} + \frac{1}{2}\sum_n s(n)e^{j2\pi(f+f_0)n}$$

$$= \frac{1}{2}S\left(e^{j2\pi(f-f_0)}\right) + \frac{1}{2}S\left(e^{j2\pi(f+f_0)}\right)$$

Question 5

Questions 2-5 pertain to the DTFT of the following signal.

$$x(n) = s(n)\cos(2\pi f_0 n)$$

where $s(n)$ is some sequence having Fourier transform $S(e^{j2\pi f})$. The answer is of the form $X(e^{j2\pi f}) = c_1 S(a_1) + c_2 S(a_2)$, where c_1, c_2 are constants and a_1, a_2 are functions of f. For example, if the first term were $S(e^{j2\pi f})$, $c_1 = 1$ and $a_1 = e^{j2\pi f}$. The first term corresponds to the *smallest* frequency component, the second term the largest.

What is a_2? To enter f_0, type `f0`.

You entered:

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Your Answer		Score	Explanation
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Question Explanation

Using the definition of the DTFT we get

$$X(e^{j2\pi f}) = \sum_n s(n)\cos(2\pi f_0 n)e^{-j2\pi f n} = \sum_n s(n)\frac{1}{2}\left[e^{j2\pi f_0 n} + e^{-j2\pi f_0 n}\right]e^{j2\pi f n}$$

$$= \frac{1}{2}\sum_n s(n)e^{j2\pi(f-f_0)n} + \frac{1}{2}\sum_n s(n)e^{j2\pi(f+f_0)n}$$

$$= \frac{1}{2}S\left(e^{j2\pi(f-f_0)}\right) + \frac{1}{2}S\left(e^{j2\pi(f+f_0)}\right)$$

Question 6

$s(n)$ is some sequence having Fourier transform $S(e^{j2\pi f})$. Find the Fourier transform of the following sequence derived from $s(n)$.

$$x(n) = \begin{cases} s\left(\frac{n}{2}\right) & \text{if } n \text{ (even)} \\ 0 & \text{if } n \text{ (odd)} \end{cases}$$

$X(e^{j2\pi f}) = S(\_\_)$. Provide the expression that fills in the blank for this equation.

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Question Explanation

Using the definition of the DTFT:

$$X(e^{j2\pi f}) = \sum_n x(n)e^{-j2\pi fn} = \sum_{n \text{ even}} s\left(\frac{n}{2}\right)e^{-j2\pi fn}$$

If we let $n = 2m,\ m = \ldots, -1, 0, 1, \ldots$ then we can rewrite the summation as

$$X(e^{j2\pi f}) = \sum_{n \text{ even}} s\left(\frac{n}{2}\right)e^{-j2\pi fn} = \sum_m s(m)e^{-j2\pi f \cdot 2m} = S(e^{j2\pi \cdot 2f})$$

So frequency response of $x(n)$ is twice as fast as that of $s(n)$.

Question 7

Questions 7-8 pertain to the DTFT of the following signal derived from $s(n)$, a signal having Fourier transform $S(e^{j2\pi f})$.

$$x(n) = ns(n)$$

The answer has the form $X(e^{j2\pi f}) = ?S(?)$.

The answer has the form $X(e^{j2\pi f}) = \_\_S(?)$. Fill in the blank.

NOTE: You may need to enter a derivative operator (e.g. $\frac{d}{df}$). If so, type `d/df`. So, if the answer is $X(e^{j2\pi f}) = 2\frac{d}{df}S(e^{j2\pi f})$, enter `2*d/df`.

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Question Explanation

Using the definition of the DTFT we notice

$$X(e^{j2\pi f}) = \sum_n ns(n)e^{-j2\pi fn}$$

If we were to rearrange the above equation to

$$X(e^{j2\pi f}) = \sum_n s(n)ne^{-j2\pi fn}$$

we would have the term $ne^{-j2\pi fn}$ which is very nearly the derivative of $e^{-j2\pi fn}$. So replacing $ne^{-j2\pi fn}$ by $\frac{1}{-j2\pi}\frac{d}{df}e^{-j2\pi fn}$ we get

$$X(e^{j2\pi f}) = \sum_n s(n)\frac{1}{-j2\pi}\frac{d}{df}e^{-j2\pi fn}$$

and using the linearity of the derivative we can remove the constant and derivative outside of the summation leaving

$$X(e^{j2\pi f}) = \frac{1}{-j2\pi}\frac{d}{df}\sum_n s(n)e^{-j2\pi fn} = \frac{1}{-j2\pi}\frac{d}{df}S(e^{j2\pi f}) = \frac{j}{2\pi}\frac{d}{df}S(e^{j2\pi f})$$

Question 8

Questions 7-8 pertain to the DTFT of the following signal derived from $s(n)$, a signal having Fourier transform $S(e^{j2\pi f})$.

$$x(n) = ns(n)$$

The answer has the form $X(e^{j2\pi f}) = ?S(?)$.

$X(e^{j2\pi f}) = ?S(\_\_)$. Fill in the blank in the equation. **NOTE:** to enter a derivative operator (e.g. $\frac{d}{df}$), type `d/df`.

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Question 9

Spectra of Finite-Duration Signals

Find the discrete-time Fourier transform of the following signal:

$$s(n) = \begin{cases} \cos^2\left(\frac{\pi}{4} n\right) & \text{if } n = \{-1, 0, 1\} \\ 0 & \text{if otherwise} \end{cases}$$

$S(e^{j2\pi f}) = ?$ (If possible, simplify your answer using trigonometric identities for sinusiods.)

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Your Answer	Score	Explanation
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Question Explanation

Evaluating $s(n)$ for the valid values of n yields

$$\cos^2\left(\frac{\pi}{4} n\right),\ n = -1, 0, 1 \ \Rightarrow s(n) = \begin{cases} \frac{1}{2} & n = -1, 1 \\ 1 & n = 0 \end{cases}$$

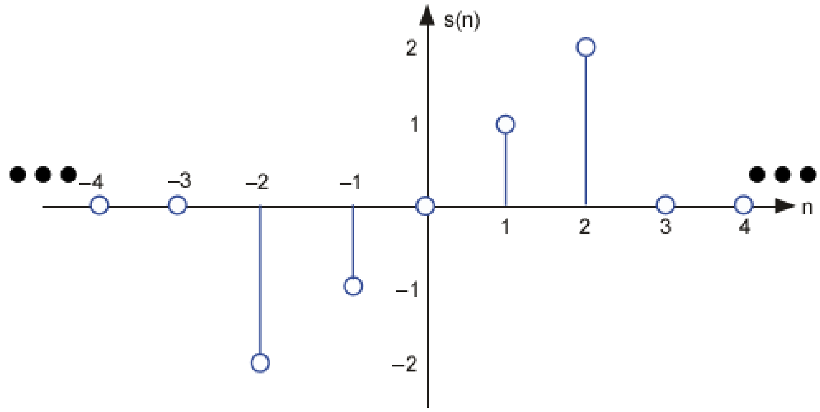
Inserting these values into the definition of the DTFT reduces the infinite summation to a finite summation with 3 terms. The expanded form of the finite summation can be further simplified through the use of a trigonometric identity.

$$S(e^{j2\pi f}) = \frac{1}{2} e^{j2\pi f} + 1 + \frac{1}{2} e^{-j2\pi f}$$

$$S(e^{j2\pi f}) = 1 + \cos(2\pi f)$$

Question 10

Find the discrete-time Fourier transform (DTFT) of the following signal



$S(e^{j2\pi f}) = ?$

If possible, simplify your answer using Euler's formula and trigonometric identities for sinusoids.

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Preview Help

Your Answer	Score	Explanation
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Question Explanation

Using the definition of the DTFT and recognizing that the summation consists of only 5 terms, the DTFT can be written as

$$\begin{aligned} S(e^{j2\pi 2f}) &= -2e^{j2\pi f} - e^{j2\pi f} + 0 + e^{-j2\pi f} + 2e^{-j2\pi 2f} \\ &= 2(e^{-j2\pi 2f} - e^{j2\pi 2f}) + (e^{-j2\pi f} - e^{j2\pi f}) \end{aligned}$$

Using Euler's formula, the DTFT of $s(n)$ can be expressed as

$$S(e^{j2\pi f}) = -4j\sin(2\pi * 2f) - 2j\sin(2\pi f)$$

Question 11

Find the DTFT of the following signal.

$$s(n) = \begin{cases} \sin(\frac{\pi}{4} n) & \text{if } n = \{0, \dots, 7\} \\ 0 & \text{if otherwise} \end{cases}$$

$S(e^{j2\pi f}) = ?$

If possible, simplify your answer using Euler's formula and trigonometric identities for sinusoids.

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Preview Help

Your Answer	Score	Explanation
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Question Explanation

We start by inserting values into the definition of the DTFT $S(e^{j2\pi f}) = \sum_n s(n) e^{-j2\pi f n}$

$$\begin{aligned} S(e^{j2\pi f}) &= 0 + \frac{1}{\sqrt{2}} e^{-j2\pi f} + e^{-j2\pi\cdot 2f} + \frac{1}{\sqrt{2}} e^{-j2\pi\cdot 3f} + 0 - \frac{1}{\sqrt{2}} e^{-j2\pi\cdot 5f} - 1e^{-j2\pi\cdot 6f} - \frac{1}{\sqrt{2}} e^{-j2\pi\cdot 7f} \\ &= \frac{1}{\sqrt{2}} e^{-j2\pi f} + e^{-j2\pi\cdot 2f} + \frac{1}{\sqrt{2}} e^{-j2\pi\cdot 3f} - \frac{1}{\sqrt{2}} e^{-j2\pi\cdot 5f} - 1e^{-j2\pi\cdot 6f} - \frac{1}{\sqrt{2}} e^{-j2\pi\cdot 7f} \\ &= \frac{1}{\sqrt{2}} e^{-j2\pi f}(1 + e^{-j2\pi\cdot 2f}) - \frac{1}{\sqrt{2}} e^{-j2\pi\cdot 5f}(1 + e^{-j2\pi\cdot 2f}) + e^{-j2\pi\cdot 2f} - e^{-j2\pi\cdot 6f} \\ &= \frac{2}{\sqrt{2}} \cos(2\pi f)(e^{-j2\pi\cdot 2f} - e^{-j2\pi\cdot 6f}) + (e^{-j2\pi\cdot 2f} - e^{-j2\pi\cdot 6f}) \\ &= e^{-j2\pi\cdot 4f}(1 + \sqrt{2}\cos(2\pi f)) \cdot 2j\sin(2\pi\cdot 2f) \end{aligned}$$

This final expression can be rearranged and written as

$$S(e^{j2\pi f}) = 2je^{-j2\pi\cdot 4f}(1 + \sqrt{2}\cos(2\pi f)) \cdot \sin(2\pi\cdot 2f).$$

Question 12

Find the length-8 **DFT** (discrete Fourier transform) of the previous signal:

$$s(n) = \begin{cases} \sin(\frac{\pi}{4} n) & \text{if } n = \{0, \ldots, 7\} \\ 0 & \text{if otherwise} \end{cases}$$

$S(k) = ?$

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Your Answer		Score	Explanation
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Question Explanation

Writing the equation for the DFT we get

$$S(k) = \sum_{n=0}^{7} s(n) e^{-j\frac{2\pi nk}{8}}.$$

Of course, the DFT is equivalent to evaluating the DTFT from the previous problem at the frequencies $f = \frac{k}{8}$.

$$\begin{aligned} S(k) &= S(e^{j2\pi f})\Big|_{f=\frac{k}{8}} \\ &= 2je^{-j2\pi\frac{k}{2}} \left(1 + \sqrt{2}\cos\left(2\pi\frac{k}{8}\right)\right) \cdot \sin\left(2\pi\frac{k}{4}\right) \end{aligned}$$

Evaluating for $k = 0, \ldots, 7$, we arrive at the expression

$$S(k) = 2j(-1)^k \left(1 + \sqrt{2}\cos\frac{\pi k}{4}\right) \sin\frac{\pi k}{2}$$

Notice that $S(k) = 0$ for $k = 0, 2, 4, 6$.

Question 13

DSP Tricks

Sammy is faced with computing lots of discrete Fourier transforms. He will, or course, use the FFT algorithm but he is behind schedule and needs to get his results as quickly as possible. He gets the idea of computing two transforms at one time by computing the transform of $s(n) = s_1(n) + js_2(n)$, where $s_1(n)$ and $s_2(n)$ are two real-valued signals for which he needs to compute the spectra. The issue is whether he can retrieve the individual DFTs $S_1(k)$ and $S_2(k)$ from the result $S(k)$ or not.

What will be the DFT $S(k)$ of this complex-valued signal in terms of $S_1(k)$ and $S_2(k)$, the DFTs of the original signals?

Express your answer in terms of S_1 and S_2 (the (k) is implied), typed as `S1` and `S2`. For example, if the answer were $S(k) = S_1(k)/5 \cdot S_2(k)$ you would type `(S1/5) * S2` as the answer.

$S(k) = ?$

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Preview Help

Your Answer		Score	Explanation
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Question Explanation

Because the DFT is linear

$$S(k) = S_1(k) + jS_2(k).$$

Question 14

Sammy's friend, who is also learning about signal processing, says that retrieving the wanted DFTs is easy: "Just find the real and imaginary parts of $S(k)$." Show that this approach is too simplistic by expressing the real and imaginary parts of $S(k)$.

Express your answer in terms of real and imaginary components of $S_1(k)$ and $S_2(k)$.

NOTE: To write the real part of $S_1(k)$, which can also be expressed as $\text{Re}[S_1(k)]$, type `ReS_1`. To write the imaginary part of $S_1(k)$ type `ImS_1`. For example, the complex function $S_1(k) = \text{Re}[S_1(k)] + j\,\text{Im}[S_1(k)]$ can be expressed by typing: `ReS_1 +`

`j*ImS_1`.

What is the real part of $S(k)$?

Express your answer in terms of S_1 and S_2 (the (k) is implied), typed as `S1` and `S2`. To enter $\text{Re}[S_1(k)]$ or $\text{Im}[S_1(k)]$, you would type `re(S1)` or `im(S1)`.

Re$[S(k)] =?$

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Your Answer		Score	Explanation
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Question Explanation

Taking the DFT of the input signal $s(n) = s_1(n) + js_2(n)$ flips the signs of the DFT of $s_2(n)$.

$$\mathcal{F}\{s_1(n)\} = S_1(k) = \text{Re}[S_1(k)] + j\,\text{Im}[S_1(k)]$$
$$\mathcal{F}\{s_2(n)\} = S_2(k) = \text{Re}[S_2(k)] + j\,\text{Im}[S_2(k)]$$

Therefore

$$S(k) = S_1(k) + jS_2(k) = \text{Re}[S_1(k)] + j\,\text{Im}[S_1(k)] + j(\text{Re}[S_2(k)] + j\,\text{Im}[S_2(k)]),$$
$$S(k) = \text{Re}[S_1(k)] + j\,\text{Im}[S_1(k)] + j\,\text{Re}[S_2(k)] - \text{Im}[S_2(k)]$$

Simply taking the real part of $S(k)$ fails to recover $S_1(k)$ because Re$[S(k)] =$ Re$[S_1(k)] -$ Im$[S_2(k)]$.

Question 15

This is a continuation of the previous question.

What is the Imaginary component of $S(k)$?

Express your answer in terms of S_1 and S_2 (the (k) is implied), typed as `S1` and `S2`. To enter $\text{Re}[S_1(k)]$ or $\text{Im}[S_1(k)]$, you would type `re(S1)` or `im(S1)`.

Im$[S(k)] =?$

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Question Explanation

Following from the answer to the previous question, $\text{Im}[S(k)] = \text{Im}[S_1(k)] + \text{Re}[S_2(k)]$.

Because $S_1(k)$ and $S_2(k)$ are complex valued, recovering $S_1(k)$ and $S_2(k)$ is not as simple as taking the real and imaginary components of $S(k)$.

Question 16

While Sammy's friend wasn't able to recover $S_1(k)$ and $S_2(k)$, his suggestion did give Sammy an idea. By using symmetry properties of the DFT Sammy was able to recover the Fourier transforms of both signals.

Sammy remembered that $\text{Re}[S_1(k)]$ and $\text{Re}[S_2(k)]$ are even (i.e $\text{Re}[S_2(k)] = \text{Re}[S_2(N-k)]$) and that $\text{Im}[S_1(k)]$ and $\text{Im}[S_2(k)]$ are odd (i.e. $\text{Im}[S_2(k)] = -\text{Im}[S_2(N-k)]$). Sammy then evaluated the even part of $S(k)$ in the following manner.

$$\begin{aligned}\frac{\text{Re}[S(k)] + \text{Re}[S(N-k)]}{2} &= \frac{\text{Re}[S_1(k)] - \text{Im}[S_2(k)] + \text{Re}[S_1(N-k)] - \text{Im}[S_2(N-k)]}{2} \\ &= \frac{1}{2}\Big[\text{Re}[S_1(k)] - \text{Im}[S_2(k)] + \text{Re}[S_1(k)] + \text{Im}[S_2(k)]\Big] \\ &= \text{Re}[S_1(k)]\end{aligned}$$

Similarily,

$$1/2 \cdot \left[\mathrm{Im}[S(k)] - \mathrm{Im}[S(N-k)]\right] = \mathrm{Im}[S_1(k)]$$
$$1/2 \cdot \left[\mathrm{Re}[S(k)] - \mathrm{Re}[S(N-k)]\right] = -\mathrm{Im}[S_2(k)]$$
$$1/2 \cdot \left[\mathrm{Im}[S(k)] + \mathrm{Im}[S(N-k)]\right] = \mathrm{Re}[S_2(k)]$$

Sammy has sped up his FFT computation time by a factor of ?

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Question Explanation

By computing two FFTs at once, Sammy is able to speed up his computation time by a factor of 2. If the input sequences are very long, then the additional overhead required to recover the FFT of $S_1(k)$ and $S_2(k)$ becomes negligible compared to the cost of having to compute the FFT twice.