Feedback — Problem Set VIII

You submitted this homework on **Tue 26 Mar 2013 3:47 PM CDT -0500**. You got a score of **0.00** out of **11.00**. However, you will not get credit for it, since it was submitted past the deadline.

In this problem set, you will be given a total of ten attempts. We will accept late submission until the fifth day after the due date, and late submission will receive half credit. Explanations and answers to the problem set will be available after the due date. Since the homework problems will become gradually more challenging as the course proceeds, we highly recommend you to start the habit of printing out the problems and working on them with paper and pencil. Also, please be sure to read the problem statements carefully and double check your expressions before you submit.

A pdf version of this problem set is available for you to print. Note: all mathematical expressions have to be exact, even when involving constants. Such an expression is required when a function and/or a variable is required in the answer. For example, if the answer is $\sqrt{3}x$, you must type `sqrt(3)*x`, not `1.732*x` for the answer to be graded as being correct.

Question 1

We can find the input-output relation for a discrete-time filter much more easily than for analog filters. The key idea is that a discrete-time signal can be written as a weighted linear combination of unit samples.

$$x(n) = \sum_i x(i)\delta(n-i),\ \ \delta(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\delta(n)$ is the unit-sample. In this expression, $x(i)$ is the amplitude of the i^{th} signal component and $x(n)$ denotes the signal being decomposed into a superposition of unit samples.

If $h(n)$ denotes the **unit-sample response**—the output of a discrete-time linear, shift-invariant filter to a unit-sample input—what is the output of the filter?

If $\delta(n) \to h(n)\, x(n) \to ?$

Your Answer	Score	Explanation
$y(n) = \sum_i x(n)h(n-i)$		
$y(n) = \sum_i h(i)n(n-i)$		
$y(n) = \sum_i h(n)x(n-i)$		
$y(n) = \sum_i x(i)h(n-i)$		
Total	0.00 / 1.00	

Question Explanation

If $\delta(n) \to h(n)$ then $x(n) \to \sum_i x(i)h(n-i)$ as the system is linear and shift-invariant. Therefore

$$y(n) = \sum_i x(i)h(n-i).$$

Question 2

Assume that we are using an FIR filter with the unit sample response having duration $q+1$. If the input has duration D, what is the duration of the filter's output to this signal?

Type your answer as an expression involving `q` and `D`.

Duration $= ?$

You entered:

Preview Help

Your Answer	Score	Explanation
✘	0.00	Could not parse student submission
Total	0.00 / 1.00	

Question Explanation

If $x(n)$ has duration D and $h(n)$ has duration $q+1$, the "last" output will be determined by the last input value. Duration equals the largest output index having $x(D-1)$ as its gain. When $i = n-1$,

$$y(n) = x(n-1)h(n-(D-1))$$

Last index of $h(n)$ is $n = q$. Therefore the last output is when $q = n-(D-1)$ or $n = q+D-1$. Therefore duration $= q+D =$ duration(x) + duration$(h) - 1$.

Question 3

Assume that we are filtering a signal with an FIR boxcar averager: $h(n) = \frac{1}{q+1}$ for $n = \{0, \ldots, q\}$ and zero otherwise.

Let the input be a pulse of unit height and duration D. Find the filter's output when $D = \frac{q+1}{2}$, where q is an odd integer.

Type an expression for the output in terms of `n`, `q` and `D`.

For $n = 0, \ldots, D-1 : \ y(n) = ?$

You entered:

Preview Help

Your Answer	Score	Explanation
✘	0.00	Could not parse student submission
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Question Explanation

The boxcar averager is simply $h(n) = \frac{1}{q+1}$, $n = 0, \ldots, q$and the input to the system is $x(n) = 1$for $n = 0, \ldots, D-1$where $D = \frac{q+1}{2}$. So the output equals $y(n) = \sum_{i=0}^{D-1} 1 \cdot h(n-i)$ There are three regions of interest in the output, when $n < D$, when $D \le n < q+1$and when $q < n$.

$$y(n) = \frac{n+1}{q+1}, \qquad n = 0, \ldots, D-1$$
$$y(n) = \frac{D}{q+1}, \qquad n = D, \ldots, q$$
$$y(n) = \frac{D-(n-q)}{q+1}, \qquad n = q+1, \ldots, D+q+1$$

Question 4

Assume that we are filtering a signal with an FIR boxcar averager: $h(n) = \frac{1}{q+1}$ for $n = \{0, \ldots, q\}$and zero otherwise.

Let the input be a pulse of unit height and duration D. Find the filter's output when $D = \frac{q+1}{2}$, where q is an odd integer.

Type an expression for the output in terms of `n`, `q` and `D`.

For $n = D, \ldots, q: \ \ y(n) = ?$

You entered:

Preview Help

Your Answer	Score	Explanation
✘	0.00	Could not parse student submission
Total	0.00 / 1.00	

Question Explanation

The boxcar averager is simply $h(n) = \frac{1}{q+1}$, $n = 0, \dots, q$and the input to the system is $x(n) = 1$for $n = 0, \dots, D-1$where $D = \frac{q+1}{2}$. So the output equals $y(n) = \sum_{i=0}^{D-1} 1 \cdot h(n-i)$ There are three regions of interest in the output, when $n < D$, when $D \leq n < q+1$and when $q < n$.

$$y(n) = \frac{n+1}{q+1}, \qquad n = 0, \dots, D-1$$

$$y(n) = \frac{D}{q+1}, \qquad n = D, \dots, q$$

$$y(n) = \frac{D-(n-q)}{q+1}, \qquad n = q+1, \dots, D+q+1$$

Question 5

Assume that we are filtering a signal with an FIR boxcar averager: $h(n) = \frac{1}{q+1}$ for $n = \{0, \dots, q\}$and zero otherwise.

Let the input be a pulse of unit height and duration D. Find the filter's output when $D = \frac{q+1}{2}$, where q is an odd integer.

Type an expression for the output in terms of `n`, `q` and `D`.

For $n = q+1, \dots, D+q+1 : \ y(n) = ?$

You entered:

Preview Help

Your Answer		Score	Explanation
	✘	0.00	Could not parse student submission
Total		0.00 / 1.00	

Question Explanation

The boxcar averager is simply $h(n) = \frac{1}{q+1}$, $n = 0, \dots, q$and the input to the system is $x(n) = 1$for $n = 0, \dots, D-1$where $D = \frac{q+1}{2}$. So the output equals $y(n) = \sum_{i=0}^{D-1} 1 \cdot h(n-i)$ There are three regions of interest in the output, when $n < D$, when $D \le n < q+1$and when $q < n$.

$$y(n) = \frac{n+1}{q+1}, \qquad n = 0, \dots, D-1$$

$$y(n) = \frac{D}{q+1}, \qquad n = D, \dots, q$$

$$y(n) = \frac{D-(n-q)}{q+1}, \qquad n = q+1, \dots, D+q+1$$

Question 6

A filter has an input-output relationship given by the difference equation

$$y(n) = \frac{1}{4}x(n) + \frac{1}{2}x(n-1) + \frac{1}{4}x(n-2).$$

What is the filter's transfer function?

$H(e^{j2\pi f}) = ?$

You entered:

Preview Help

Your Answer		Score	Explanation
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Total		0.00 / 1.00	

Question Explanation

A delay in the time domain is a shift in the frequency domain which is given by the Fourier transform pair $\mathcal{F}\{x(n-a)\} = e^{j2\pi \cdot af}$where a is an integer. Using this relationship we can write the transfer function as:

$$\begin{aligned} H(e^{j2\pi f}) &= \frac{1}{4} + \frac{1}{2}e^{-j2\pi f} + \frac{1}{8}e^{-j2\pi 2f} \\ &= e^{-j2\pi f}\left[\frac{1}{4}e^{j2\pi f} + \frac{1}{2} + \frac{1}{4}e^{-j2\pi f}\right] \\ H(e^{j2\pi f}) &= e^{-j2\pi f} \cdot \frac{1}{2}\left[\cos 2\pi f + 1\right] \end{aligned}$$

Question 7

For the same filter, which has an input-output relationship given by the difference equation,

$$y(n) = \frac{1}{4}x(n) + \frac{1}{2}x(n-1) + \frac{1}{4}x(n-2).$$

What is the filter's output when the input equals $\cos\left(\frac{\pi n}{2}\right)$?

$y(n) = ?$

You entered:

Preview Help

Your Answer	Score	Explanation
✘	0.00	Could not parse student submission
Total	0.00 / 1.00	

Question Explanation

$y(n) = \frac{1}{2}\cos(\frac{\pi}{2}(n-1))$

Substituting in $\cos(\frac{\pi \cdot n}{2})$ for $x(n)$ gives

$$\begin{aligned} y(n) &= \frac{1}{4}x(n) + \frac{1}{2}x(n-1) + \frac{1}{4}x(n-2) \\ &= \frac{1}{4}\cos\left(\frac{\pi n}{2}\right) + \frac{1}{2}\cos\left(\frac{\pi(n-1)}{2}\right) + \frac{1}{4}\cos\left(\frac{\pi(n-2)}{2}\right) \end{aligned}$$

The argument for the first term $\cos(\frac{\pi n}{2})$ and the argument for the third term $\cos(\frac{\pi(n-2)}{2})$ cancel which leaves the middle term.

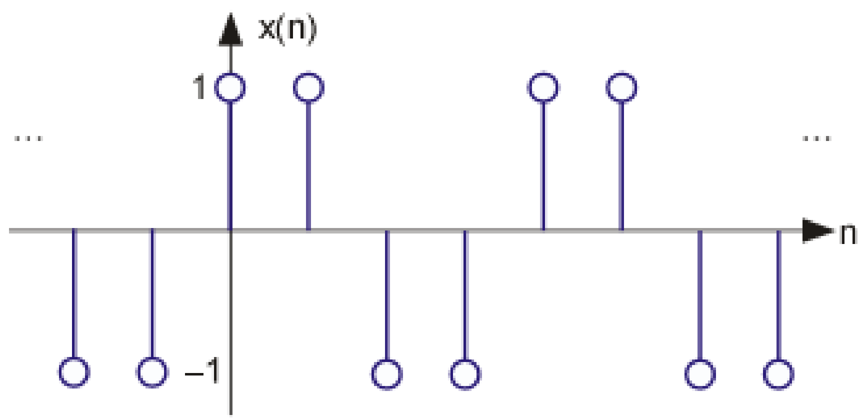
Possibly a better way to solve this problem is to use the transfer function. Express $\cos(\frac{\pi n}{2})$ as $\mathrm{Re}\left[e^{j\frac{\pi n}{2}}\right]$. The frequency is $\frac{1}{4}$, so that the output is $\mathrm{Re}\left[H\left(e^{j2\pi f}\right)\Big|_{f=\frac{1}{4}} \cdot e^{j\frac{\pi n}{2}}\right]$. At this frequency, the transfer function equals $\frac{e^{-j\pi/2}}{2} = -j/2$ Thus the output is $\mathrm{Re}\left[-\frac{j}{2}e^{j\frac{\pi n}{2}}\right] = \frac{1}{2}\sin\left(\frac{\pi n}{2}\right)$, which does equal the middle term.

Question 8

A filter has an input-output relationship given by the difference equation

$$y(n) = \frac{1}{4}x(n) + \frac{1}{2}x(n-1) + \frac{1}{4}x(n-2).$$

What is the filter's output when the input is the depicted discrete-time square wave?



Your answer should be an expression involving well-known signals.

$y(n) = ?$

You entered:

Preview Help

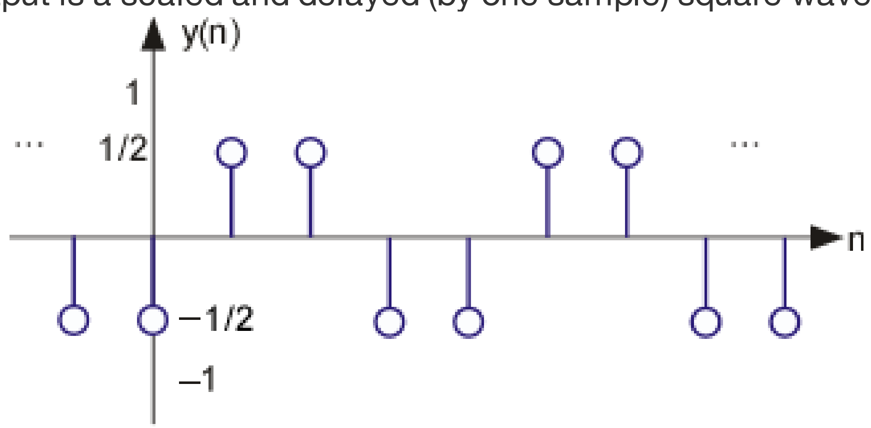
Your Answer	Score	Explanation
✘	0.00	Could not parse student submission
Total	0.00 / 1.00	

Question Explanation

Note that starting at $n = 0, \cos\left(\frac{\pi n}{2}\right) = 1, 0, -1, 0, 1, 0, -1, 0, \ldots$
Thus, the input can be expressed as $x(n) = \cos(\frac{\pi n}{2}) + \cos(\frac{\pi(n-1)}{2}.)$By direct substitution and using superposition and the result of the previous question, the output is

$$y(n) = \frac{1}{2}\left[\cos(\frac{\pi(n-1)}{2}) + \cos(\frac{\pi(n-2)}{2})\right] = \frac{1}{2}\left[\sin\frac{\pi n}{2} + \sin\frac{\pi(n-1)}{2}\right].$$

The output is a scaled and delayed (by one sample) square wave.



Question 9

A discrete-time system is governed by the difference equation

$$y(n) = y(n-1) + \frac{x(n) + x(n-1)}{2}.$$

Find the transfer function for this system. Simplify your answer as much as possible.

$H(e^{j2\pi f}) = ?$

You entered:

Preview Help

Your Answer	Score	Explanation
✘	0.00	Could not parse student submission
Total	0.00 / 1.00	

Question Explanation

$y(n) = y(n-1) + \frac{x(n)+x(n-1)}{2}$. Using a Fourier transform property, we know that a delay in the time domain corresponds to a shift in the frequency domain, and using the definition of the transfer function $H(e^{j2\pi f}) = \frac{Y(e^{j2\pi f})}{X(e^{j2\pi f})}$ we find

$$\begin{aligned}
\mathcal{F}\{y(n) - y(n-1)\} &= \mathcal{F}\left\{\frac{1}{2}\left(x(n) + x(n-1)\right)\right\} \\
Y(e^{j2\pi f}) - Y(e^{j2\pi f})e^{j2\pi f} &= \frac{1}{2}\left(X(e^{j2\pi f}) + X(e^{j2\pi f})e^{j2\pi f}\right) \\
Y(e^{j2\pi f})\left[1 - e^{j2\pi f}\right] &= \frac{1}{2}X(e^{j2\pi f})\left[1 + e^{j2\pi f}\right]
\end{aligned}$$

Therefore,

$$H(e^{j2\pi f}) = \frac{Y(e^{j2\pi f})}{X(e^{j2\pi f})} = \frac{\frac{1}{2}\left[1 + e^{-j2\pi f}\right]}{1 - e^{-j2\pi f}} = \frac{1}{2}\frac{\cos(\pi f)}{j\sin(\pi f)} = \frac{-j}{2}\cot(\pi f)$$

Question 10

An electronics startup wants to develop a filter that would be used in analog applications, but that is implemented digitally. The filter is to operate on signals that have a 10 kHz bandwidth, and will serve as a lowpass filter.
What minimum sampling rate (in kilohertz) must be used and how many bits must be used in the A/D converter for the acquired signal's signal-to-noise ratio to be at least 60 dB? (For the SNR calculation, assume the signal is a sinusoid.)

Enter your answer as a list. **Note:** the frequency has units of kHz for this problem. For example, if the necessary sampling rate was 1 kHz and the A/D converter needed 4 bits, you would type `1 4`

You entered:

Your Answer		Score	Explanation
	✘	0.00	
Total		0.00 / 1.00	

Question Explanation

By Shannon's sampling theorem, we know that the sampling rate must be at least $F_s = 20$ kHz.

The SNR is found using the equation for a sinusoid. SNR $= \frac{3}{2} 2^{2B}$ or $6B + 10\log_{10} 15 = 60$ where $10\log_{10} 15 = 1.76$. Therefore, $B \geq \frac{60-1.76}{6} = 9.71$, so we need $B = 10$ in order to reach an SNR of 60.

Question 11

Another consideration for the startup company's design is how to implement the filter. If the filter is a length-128 FIR filter (the duration of the filter's unit-sample response equals 128), should it be implemented in the time or frequency domain?

Your Answer	Score	Explanation
Implement directly in the time domain		
Implement in the frequency domain using FFT		
Total	0.00 / 1.00	

Question Explanation

The filter's unit sample response has a duration of 128, we should *definitely* use the FFT and implement in the frequency domain.