Fundamentals of Electrical Engineering

Error Correcting Codes

- Correcting errors

Digital Communication Model

Source coder: $\mathbf{c} = \mathbf{G}\mathbf{b}$ $\quad d_{\min} \geq 3$

Example: (7,4) Hamming code

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hdashline 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

How to Correct Errors?

Source coder: $\mathbf{c} = \mathbf{G}\mathbf{b}$

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hdashline 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Channel decoder receives $\widehat{\mathbf{c}} = \mathbf{c} \oplus \mathbf{e}$

Decoder forms $\mathbf{s} = \mathbf{H}\widehat{\mathbf{c}} = \mathbf{H}(\mathbf{c} \oplus \mathbf{e}) = \mathbf{H}\mathbf{c} \oplus \mathbf{H}\mathbf{e}$

$$= \mathbf{H}\mathbf{G}\mathbf{b} \oplus \mathbf{H}\mathbf{e}$$

$$= \mathbf{H}\mathbf{e}$$

$$\mathbf{H} = \left[\begin{array}{cccc:ccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

lower portion of $\mathbf{G}$ | identity

RICE

How to Correct Errors?

$$\mathbf{c} = \mathbf{G}\mathbf{b} \quad \mathbf{G} = \begin{bmatrix} \mathbf{I} \\ \mathbf{G}_{\text{lower}} \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} \mathbf{b} \\ \text{- - - - -} \\ \mathbf{G}_{\text{lower}}\mathbf{b} \end{bmatrix}$$

$$\mathbf{H} = \left[\mathbf{G}_{\text{lower}} \;\vdots\; \mathbf{I}\right]$$

$$\mathbf{H}\mathbf{c} = \mathbf{H}\mathbf{G}\mathbf{b} = \left[(\mathbf{G}_{\text{lower}}\mathbf{b}) \oplus (\mathbf{G}_{\text{lower}}\mathbf{b})\right] = \mathbf{0}$$

Error Correction

1. For each received "codeword", form

$$\mathbf{H}\widehat{\mathbf{c}} = \mathbf{He}$$

2. If non-zero, find error sequence

$\mathbf{e}$	$\mathbf{He}$
1000000	101
0100000	111
0010000	110
0001000	011
0000100	100
0000010	010
0000001	001

$$\mathbf{H} = \left[\begin{array}{cccc:ccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

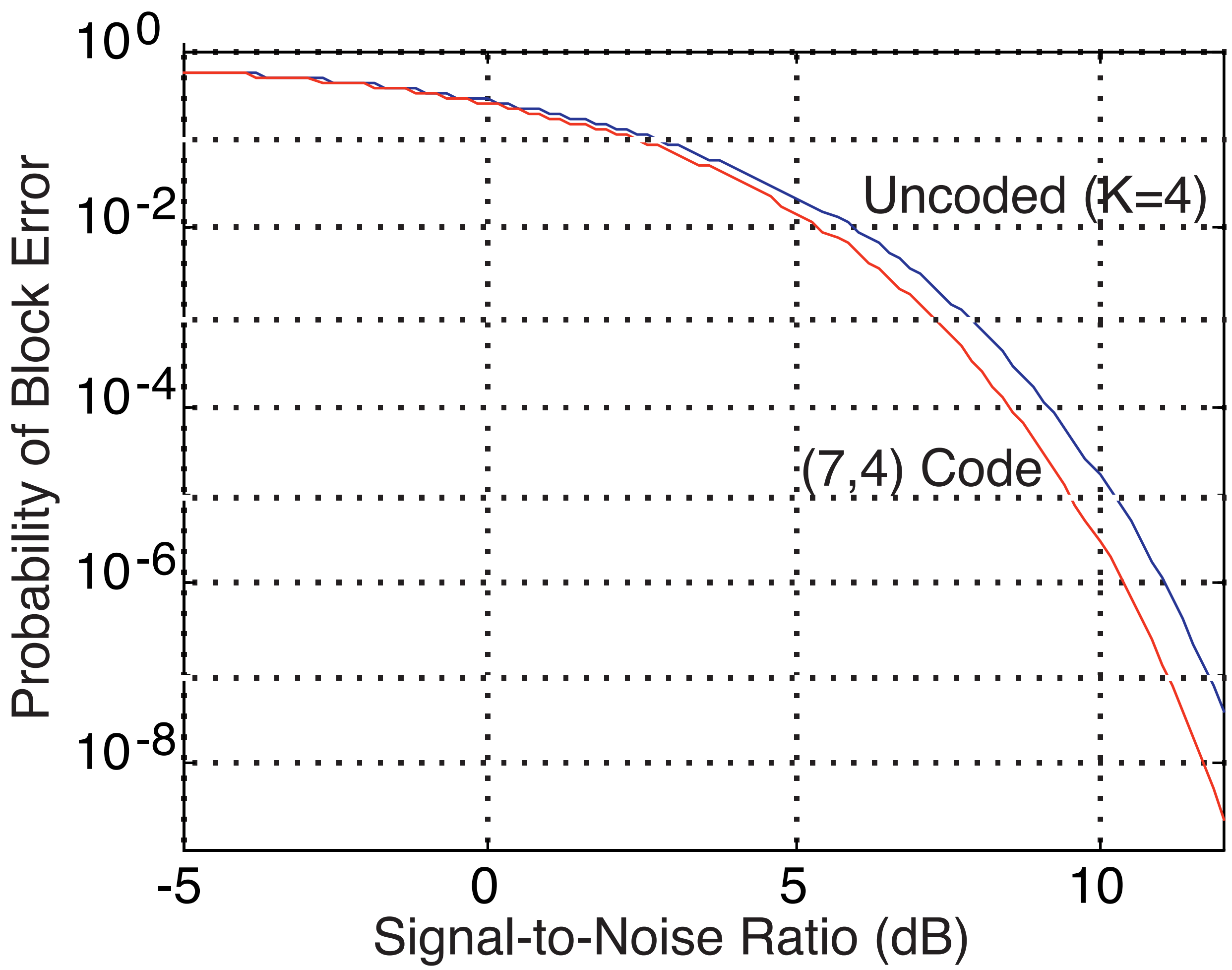
3. Add error sequence to received "codeword"

$$\widehat{\mathbf{c}} \oplus \mathbf{e} = (\mathbf{c} \oplus \mathbf{e}) \oplus \mathbf{e} = \mathbf{c}$$

4. Find data bits in corrected codeword

Do we win?

- Remember, to send data at the same rate, we must scale the bit interval duration by the coding efficiency factor $E = K/N$



Hamming Codes

- For a (N, K) code, number of single-bit errors = N
- Number of non-zero values for $\mathbf{H}\widehat{\mathbf{c}} = 2^{N-K} - 1$
- To correct any single-bit error, must have

$$2^{N-K} - 1 \geq N,\ N - K = \text{ number of coding bits}$$

- Maximally efficient when equality applies

$$2^{N-K} - 1 = N$$

N	K	E
3	1	0.33
7	4	0.57
15	11	0.73
31	26	0.84
63	57	0.90
127	120	0.94

But…

- The efficient Hamming code can correct all *single*-bit errors
- When do double-bit errors become more likely than single-bit errors?

$$\underbrace{\binom{N}{1} p_e (1-p_e)^{N-1}}_{\text{single-bit error probability}} \overset{?}{<} \underbrace{\binom{N}{2} p_e^2 (1-p_e)^{N-2}}_{\text{double-bit error probability}}$$

$$\text{double-bit errors more likely when } \frac{p_e}{1-p_e} > \frac{2}{N-1}$$

Error-Correcting Codes

- Even though the channel introduces transmission errors, error correcting codes can repair the errors
- However, it would seem that errors are always present in digital communication