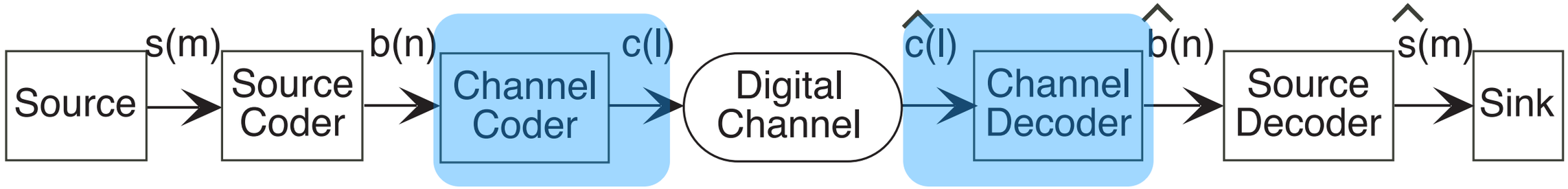
Fundamentals of Electrical Engineering

Error Correcting Codes

- Repetition codes
- Hamming codes

Digital Communication Model



- Can we mitigate channel-induced errors?
- Error correcting codes (ECC)

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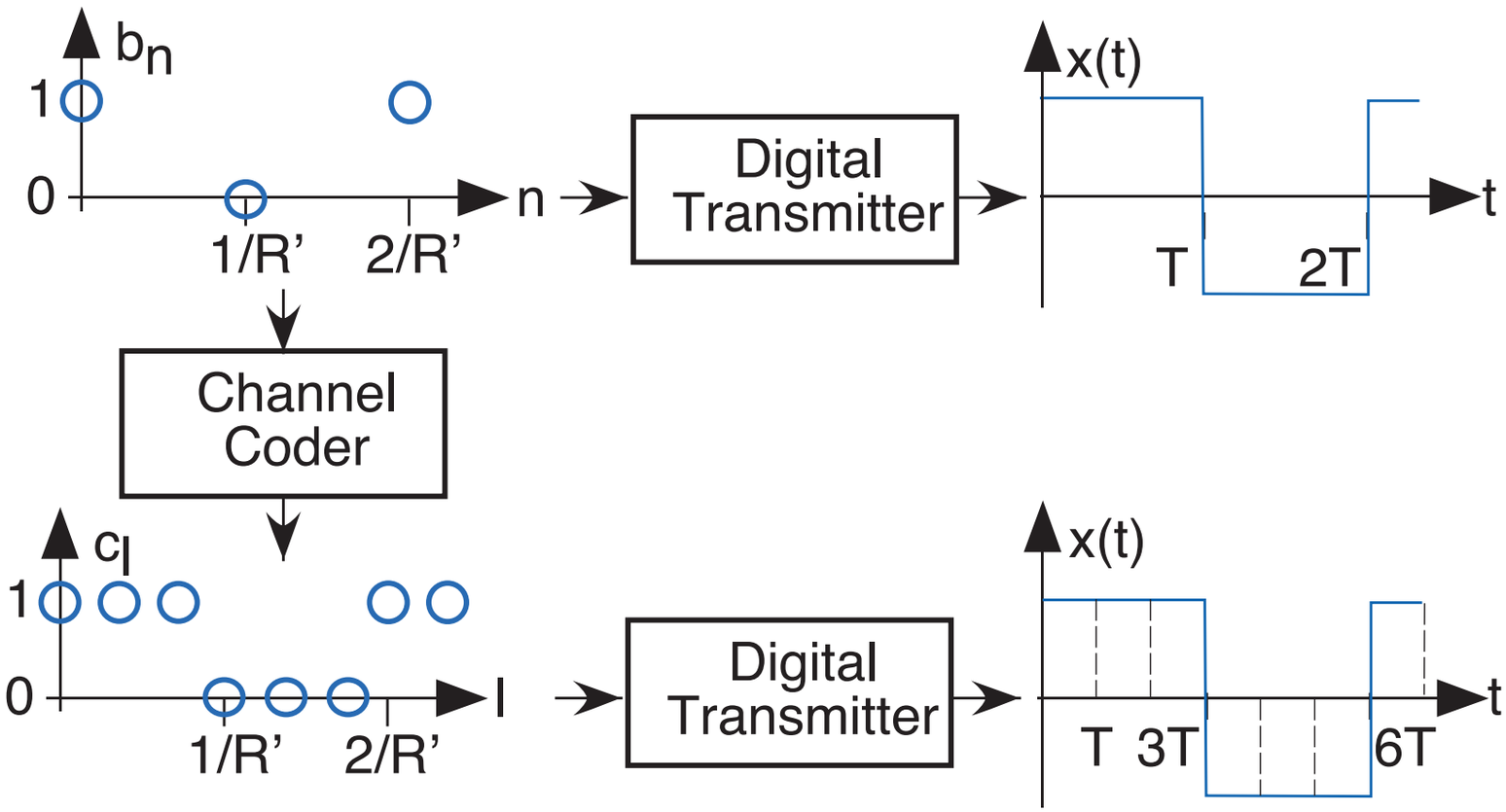
Repetition Codes

$$b(n) \leftrightarrow c(l) \qquad b(n) \leftrightarrow c(l)$$

$$0 \leftrightarrow 000 \qquad 1 \leftrightarrow 111$$

Simply repeat each source bit N times (N odd)



- Creates what is known as a (N, 1) code
- Has a *code rate* of $1/N$

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Decoding

Decode by "majority vote"

When the source bit is "0"…

Receiver Codeword	**Decoded Bit**	**Probability**
000	0 ✔	$(1-p_e)^3$
001	0 ✔	$p_e(1-p_e)^2$
010	0 ✔	$p_e(1-p_e)^2$
011	1 ✘	$p_e^2(1-p_e)$
100	0 ✔	$p_e(1-p_e)^2$
101	1 ✘	$p_e^2(1-p_e)$
110	1 ✘	$p_e^2(1-p_e)$
111	1 ✘	p_e^3

$$\Pr[\text{decoding error}] = 3p_e^2(1-p_e) + p_e^3$$

$$\text{If } p_e < \frac{1}{2}, 3p_e^2(1-p_e) + p_e^3 < p_e$$

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Repetition Codes

- Good point: very simple
- Bad points
  - ✳ Low coding rate ($1/N$) for a single-bit error correcting code
  - ✳ As N increases, single-bit errors become less likely
- Need more powerful codes

Block Codes

(N,K) codes: represent a block of K source bits with N bits

Example: (7,4) code

$$c(1) = b(1)$$
$$c(2) = b(2)$$
$$c(3) = b(3)$$
$$c(4) = b(4)$$
$$c(5) = b(1) \oplus b(2) \oplus b(3)$$
$$c(6) = b(2) \oplus b(3) \oplus b(4)$$
$$c(7) = b(1) \oplus b(2) \oplus b(4)$$

exclusive-or (XOR)
binary arithmetic

$$0 \oplus 0 = 0$$
$$0 \oplus 1 = 1$$
$$1 \oplus 0 = 1$$
$$1 \oplus 1 = 0$$

Hamming Distance

$$d(c_1, c_2) = \text{sum}(c_1 \oplus c_2)$$

Example: $c_1 = [01101010]$ $\quad c_1 \oplus c_2 = [00110000]$

$c_2 = [01110010]$ $\quad d(c_1, c_2) = 2$

Note: $c_2 = c_1 \oplus [00110000]$

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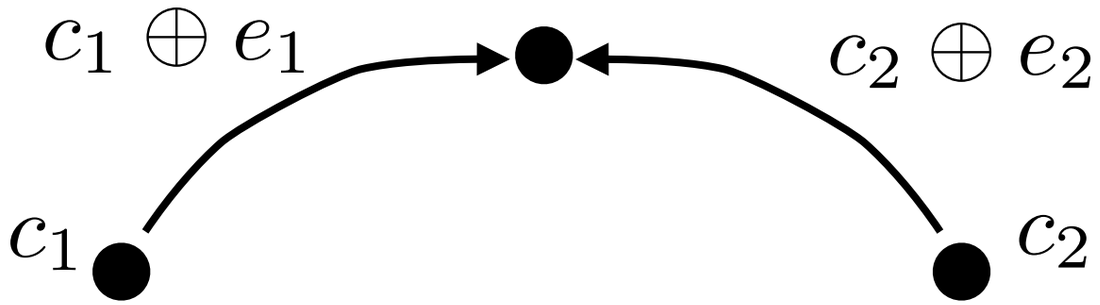
Hamming Distance

How far apart do two codewords need to be so that in spite of a single-bit error, we know what the actual codeword was?

or

When can a single-bit error in one codeword *not* be confused with a single-bit error in another?



$$\text{sum}(e_1) = 1 = \text{sum}(e_2)$$

$$c_2 = c_1 \oplus e_1 \oplus e_2 \qquad d(c_1, c_2) = 2$$

To have a single-bit error correcting code, must have

$$\min_{i \neq j} d(c_i, c_j) \geq 3$$

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Repetition Code

$$0 \leftrightarrow 000 \qquad c_1$$

$$1 \leftrightarrow 111 \qquad c_2$$

$$d(c_1, c_2) = 3 \qquad \checkmark$$

Re-express creating codewords from data bits with matrices using binary arithmetic

$$\mathbf{c} = \mathbf{G}\mathbf{b} \qquad \mathbf{b} = [b(1)] \qquad \mathbf{G} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} c(1) \\ c(2) \\ c(3) \end{bmatrix}$$

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(7,4) Code

$$
\begin{aligned}
c(1) &= b(1) \\
c(2) &= b(2) \\
c(3) &= b(3) \\
c(4) &= b(4) \\
c(5) &= b(1) \oplus b(2) \oplus b(3) \\
c(6) &= b(2) \oplus b(3) \oplus b(4) \\
c(7) &= b(1) \oplus b(2) \oplus b(4)
\end{aligned}
$$

$$
\mathbf{b} = \begin{bmatrix} b(1) \\ b(2) \\ b(3) \\ b(4) \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} c(1) \\ c(2) \\ c(3) \\ c(4) \\ c(5) \\ c(6) \\ c(7) \end{bmatrix}
$$

$$
\mathbf{c} = \mathbf{G}\mathbf{b}
$$

$$
\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}
$$

Properties of Block Codes

Because $\mathbf{c} = \mathbf{G}\mathbf{b}$, code is *linear*

$$\mathbf{c} = \mathbf{G}(\mathbf{b}_1 \oplus \mathbf{b}_2) = \mathbf{c}_1 \oplus \mathbf{c}_2$$

Consequently, "adding" two codewords yields a codeword

Distance between any two codewords equals the number of bits in another codeword

$$d(\mathbf{c}_1, \mathbf{c}_2) = \mathbf{c}_1 \oplus \mathbf{c}_2 = \text{sum}(\mathbf{c}_3)$$

Need only examine individual codewords to determine the code's error correcting capability

Back to (7,4) Code

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$$3 \quad 4 \quad 3 \quad 3$$

So we have a single-bit error correcting code!

And the code rate (efficiency) = 4/7

Error-Correcting Codes

- Even though the channel introduces transmission errors, error correcting codes can repair some of the errors
- Very different than analog communication
- Must send bits at a higher rate than required by the source, but efficient codes are easily designed
- However, ECCs have limited capabilities

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