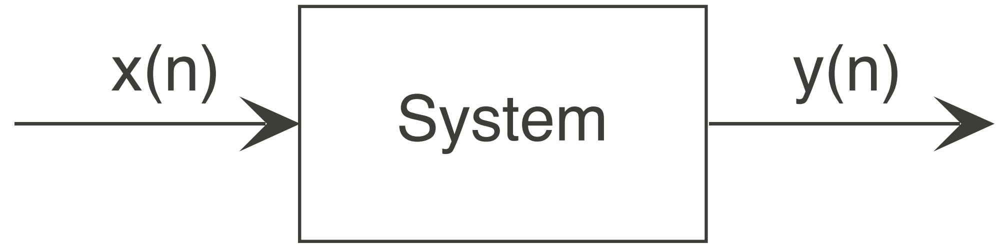
Fundamentals of Electrical Engineering

Digital Filters

- Difference equations
- Filter categories
- Input-output relationships in time and frequency

RICE

Difference Equations

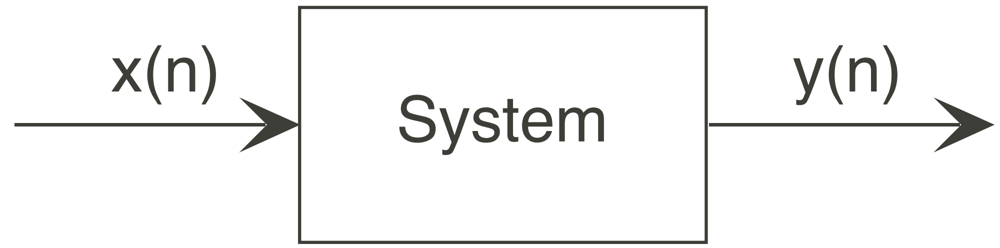


- How do we build filters for digital signals?

$$y(n) = a_1 y(n-1) + \cdots + a_p y(n-p) \\ + b_0 x(n) + b_1 x(n-1) + \cdots + b_q x(n-q)$$

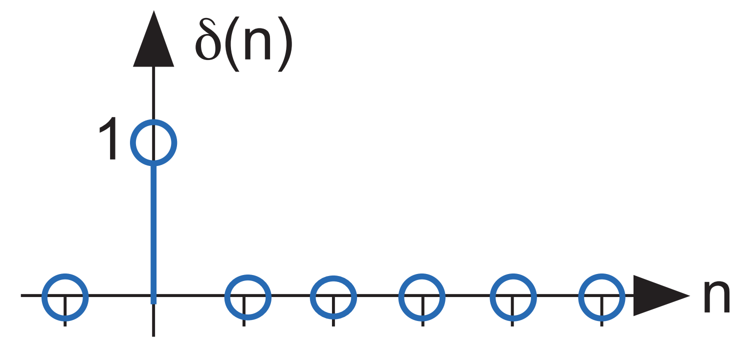
- Explicit input-output formula

Example Difference Equation



$$y(n) = ay(n-1) + bx(n), \quad x(n) = \delta(n)$$

n	$x(n)$	$y(n)$
-1	0	0
0	1	b
1	0	$b \bullet a$
2	0	$b \bullet a^2$
$\ldots$	0	$\ldots$
n	0	$b \bullet a^n$

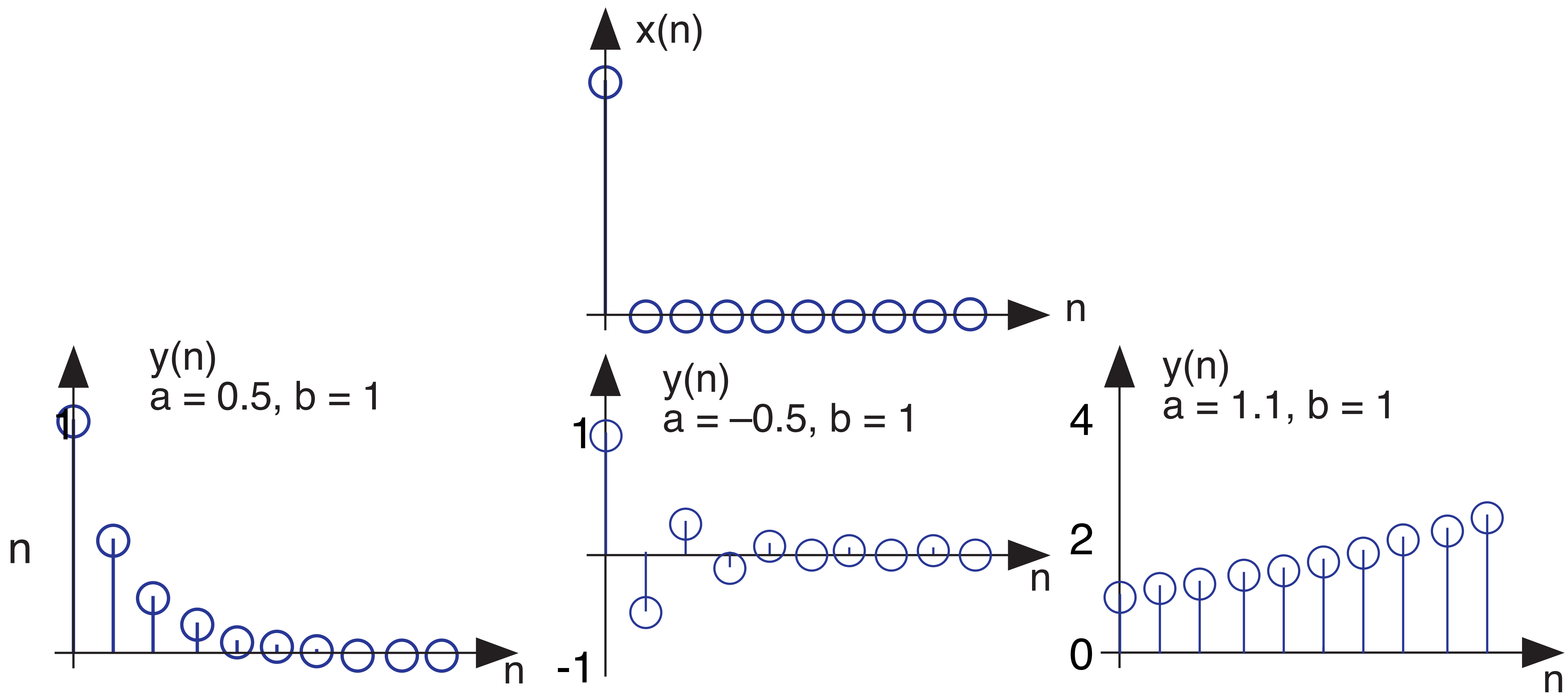


$$y(n) = b \cdot a^n \mathrm{u}(n)$$

Example Difference Equation

$$y(n) = ay(n-1) + bx(n), \quad x(n) = \delta(n)$$

$$y(n) = b \cdot a^n \mathrm{u}(n)$$



Difference Equations in Frequency Domain

$$y(n) = ay(n-1) + bx(n), \quad x(n) = Xe^{j2\pi fn}$$

Assume $y(n) = Ye^{j2\pi fn}$

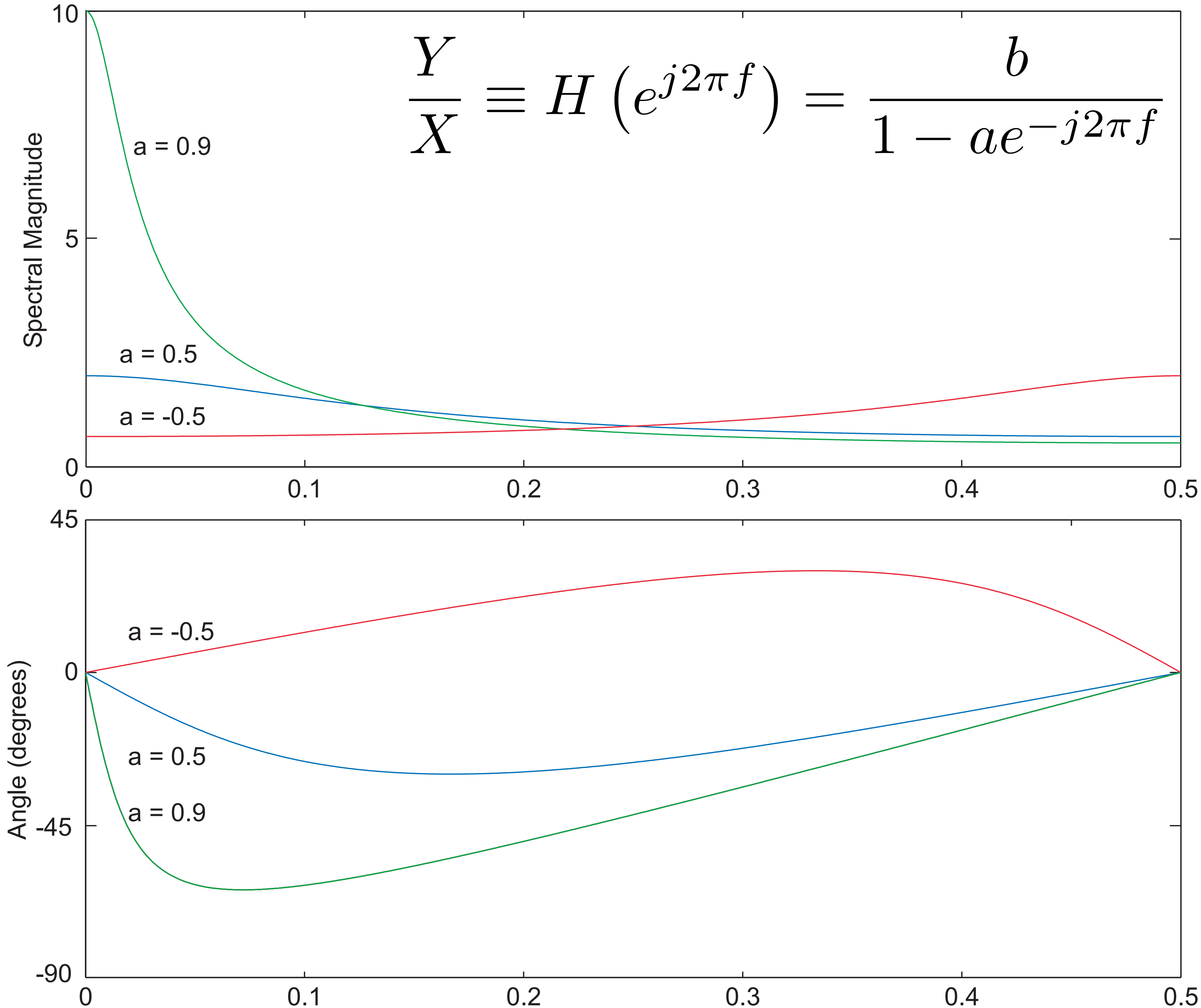
Note that $y(n-1) = Ye^{j2\pi f(n-1)} = Ye^{-j2\pi f}e^{j2\pi fn}$

$$Y\cancel{e^{j2\pi fn}} = aYe^{-j2\pi f}\cancel{e^{j2\pi fn}} + bX\cancel{e^{j2\pi fn}}$$

$$\Longrightarrow \frac{Y}{X} \equiv H\left(e^{j2\pi f}\right) = \frac{b}{1 - ae^{-j2\pi f}}$$

$$y(n) = H\left(e^{j2\pi f}\right) \cdot Xe^{j2\pi fn}$$

Transfer Function Plots



Difference Equations in Frequency Domain

$$y(n) = a_1 y(n-1) + \cdots + a_p y(n-p) + b_0 x(n) + b_1 x(n-1) + \cdots + b_q x(n-q)$$

By assuming $x(n) = Xe^{j2\pi fn}$

$$\frac{Y}{X} \equiv H\left(e^{j2\pi f}\right) = \frac{b_0 + b_1 e^{-j2\pi f} + \cdots + b_q e^{-j2\pi fq}}{1 - a_1 e^{-j2\pi f} - \cdots - a_p e^{-j2\pi fp}}$$

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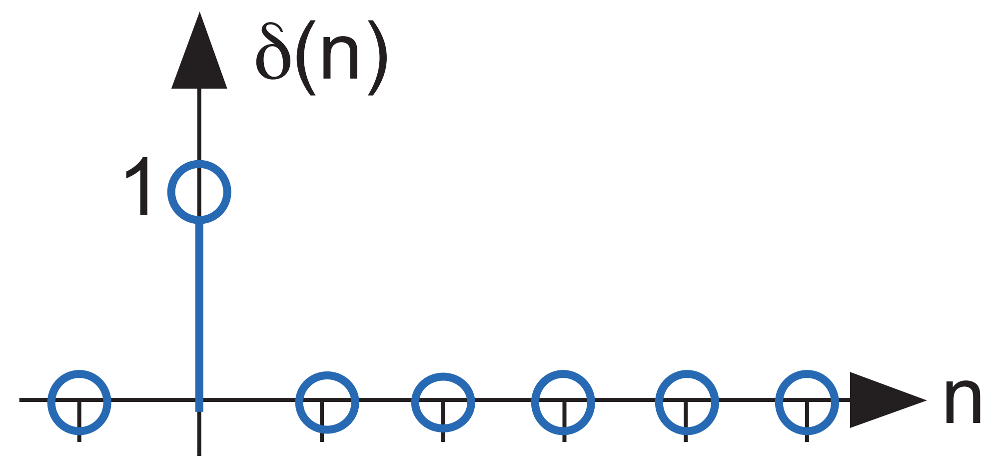
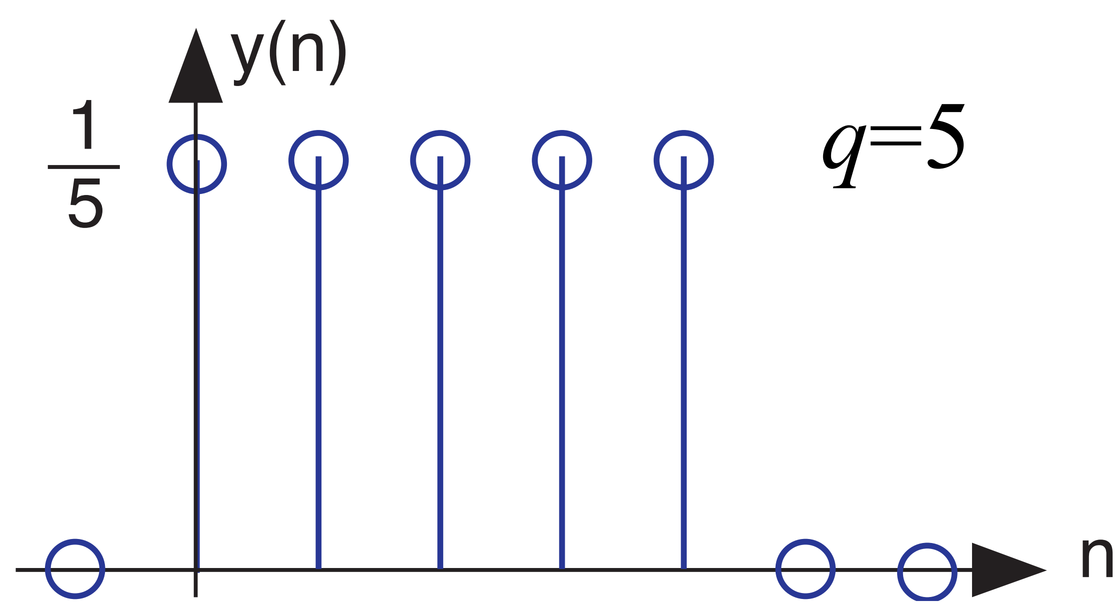
A Special Filter

$$y(n) = \frac{1}{q}\left[x(n) + x(n-1) + \cdots + x(n-q+1)\right]$$

Letting $x(n) = \delta(n)$

n	$x(n)$	$y(n)$
–1	0	0
0	1	$1/q$
1	0	$1/q$
2	0	$1/q$
…	0	…
q-1	0	$1/q$
q	0	0
…	0	0



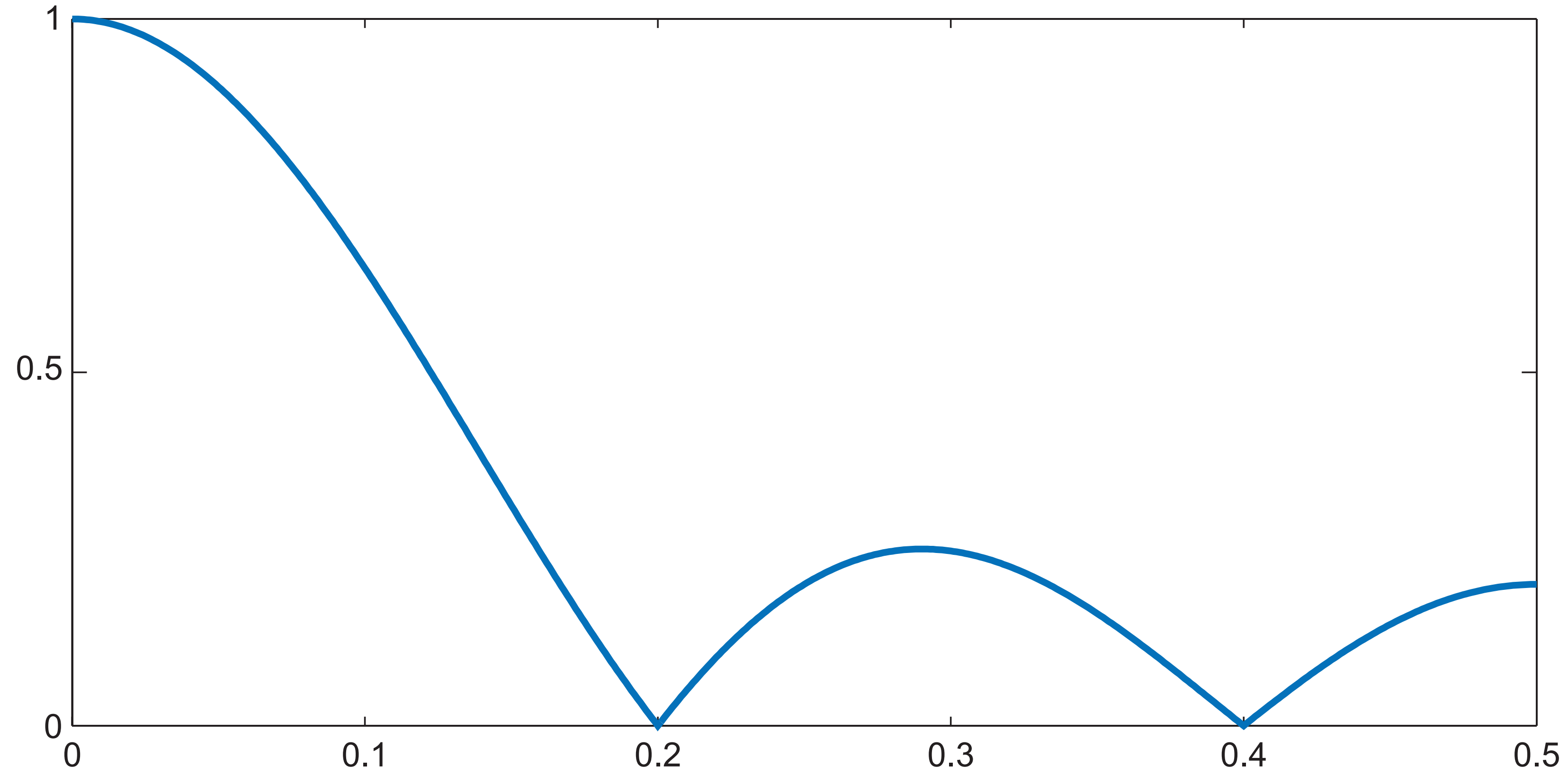


q=5

A Special Filter

$$y(n) = \frac{1}{q}\left[x(n) + x(n-1) + \cdots + x(n-q+1)\right]$$

$$H\left(e^{j2\pi f}\right) = e^{-j\pi f(q-1)} \frac{\sin \pi f q}{q \sin \pi f}$$



Transfer Function and Unit-Sample Response

$$\delta(n) \rightarrow h(n)$$

$$\delta(n-m) \rightarrow h(n-m)$$

$$x(n) = \sum_{m=-\infty}^{\infty} x(m)\delta(n-m) \longrightarrow y(n) = \sum_{m=-\infty}^{\infty} x(m)h(n-m)$$

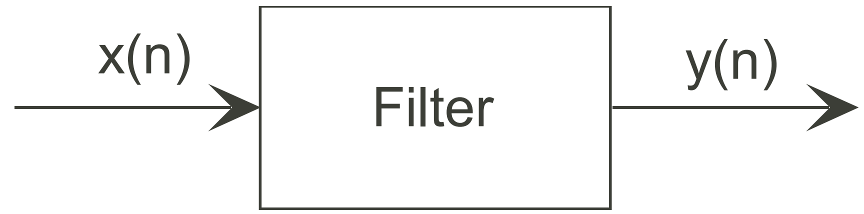
$$\sum_{n=-\infty}^{\infty} y(n)e^{-j2\pi fn} = \sum_{m=-\infty}^{\infty} x(m) \sum_{n=-\infty}^{\infty} h(n-m)e^{-j2\pi fn}$$

$$= \sum_{m=-\infty}^{\infty} x(m)H(e^{j2\pi f})e^{-j2\pi fm}$$

$$\boxed{Y(e^{j2\pi f}) = X(e^{j2\pi f})H(e^{j2\pi f})}$$

RICE

FIR and IIR Filters



FIR

$$y(n) = \sum_{k=0}^{q-1} b_k x(n-k)$$

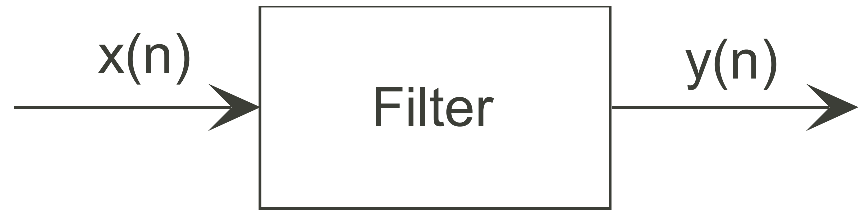
- Unit-sample response has finite duration (q)
- If $x(n)$ has duration N, output has duration N+q-1
- Can have linear phase

IIR

$$y(n) = \sum_{\ell=1}^{p} a_\ell y(n-\ell) + \sum_{k=0}^{q-1} b_k x(n-k)$$

- Unit-sample response is infinitely long
- If $x(n)$ has a finite duration, output has an infinite duration
- Always has nonlinear phase

FIR and IIR Filters



<u>FIR</u>

$$y(n) = \sum_{k=0}^{q-1} b_k x(n-k)$$

- Unit-sample response has finite duration (q)
- If $x(n)$ has duration

<u>IIR</u>

$$y(n) = \sum_{\ell=1}^{p} a_\ell y(n-\ell) + \sum_{k=0}^{q-1} b_k x(n-k)$$

- Unit-sample response is infinitely long
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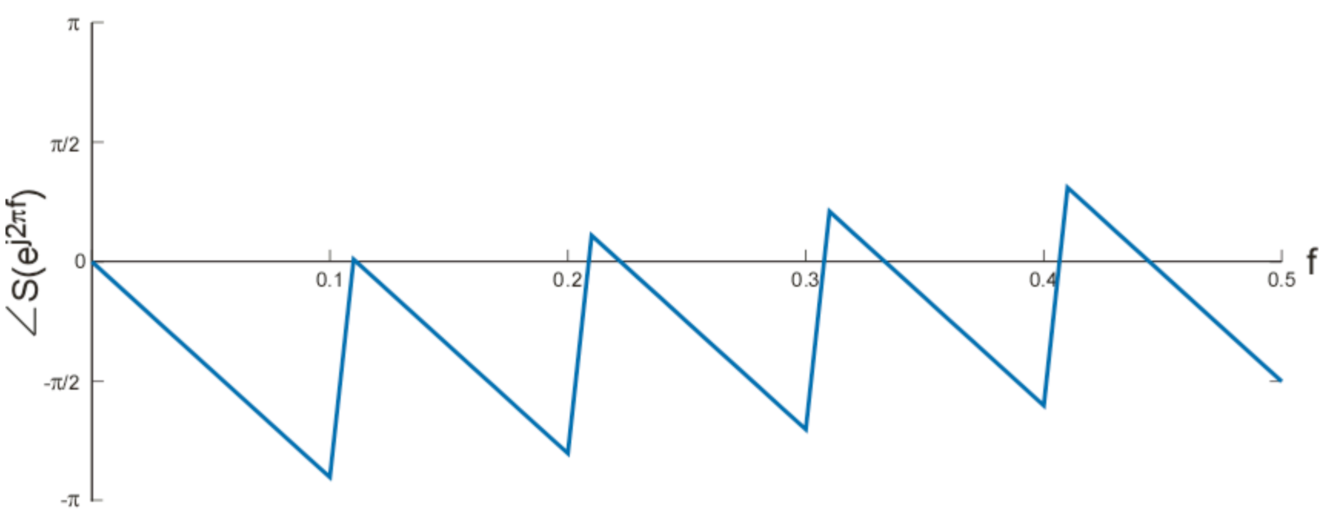

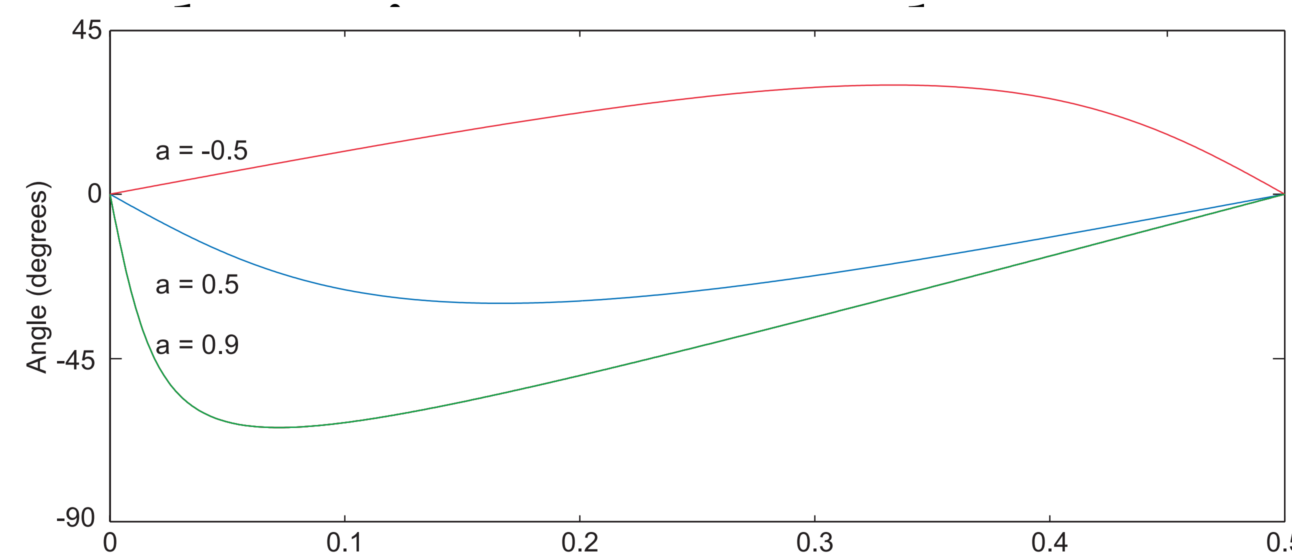

Digital Filters

- Interesting and important filters are described by difference equations
- The difference equation provides a method, not always the most efficient, of implementing (programming) the filter
- The input-output relationship for a linear, shift-invariant filter can be expressed in the time domain or the frequency domain

$$y(n) = \sum_{m=0}^{N-1} x(m)h(n-m) \quad Y(e^{j2\pi f}) = X(e^{j2\pi f})H(e^{j2\pi f})$$

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- The unit-sample response and transfer function are Fourier transform pairs

$$\boxed{h(n) \longleftrightarrow H(e^{j2\pi f})}$$