Fundamentals of Electrical Engineering

Discrete-Time Spectral Analysis

- Discrete Fourier transform (DFT)
- Properties

RICE

Computing the DTFT?

$$S(e^{j2\pi f}) = \sum_{n=-\infty}^{\infty} s(n) e^{-j2\pi f n}$$

- Two problems
  - ✳ Infinite duration signals
  - ✳ Continuous frequency variable
- Solutions
  - ✳ Finite-duration signals $s(n),\ 0 \leq n \leq N-1$
  - ✳ *Sample* frequency $f = \frac{k}{K},\ 0 \leq k \leq K-1$

Discrete Fourier Transform (DFT)

$$S(k) = \sum_{n=0}^{N-1} s(n) e^{-j\frac{2\pi nk}{K}} \quad k = 0, \dots, K-1$$

DFT transform length–K–how big should it be?

Inverse transform formula has to be of the form

$$s(n) \propto \sum_{k=0}^{K-1} S(k) e^{j\frac{2\pi nk}{K}} \quad n = 0, \dots, N-1$$

For this formula to work, need $K \geq N$

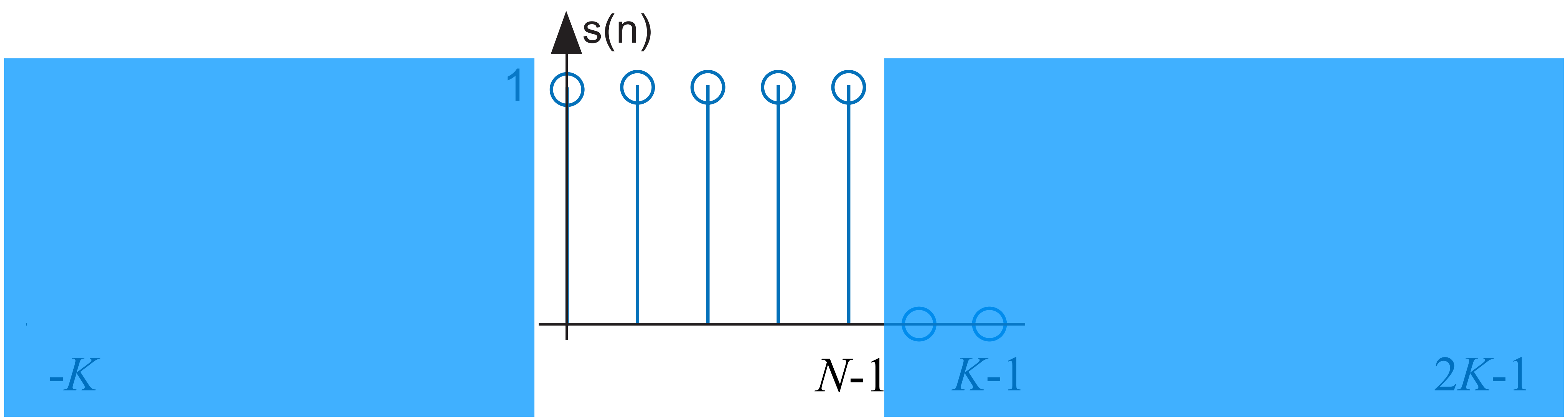
Evaluating a DFT with a length greater than the signal's duration equivalent to "padding" the original signal with zeros

DFT and IDFT

DFT $$S(k) = \sum_{n=0}^{N-1} s(n) e^{-j\frac{2\pi nk}{K}} \quad k = 0, \ldots, K-1$$

IDFT $$s(n) = \frac{1}{K} \sum_{k=0}^{K-1} S(k) e^{j\frac{2\pi nk}{K}} \quad n = 0, \ldots, N-1$$

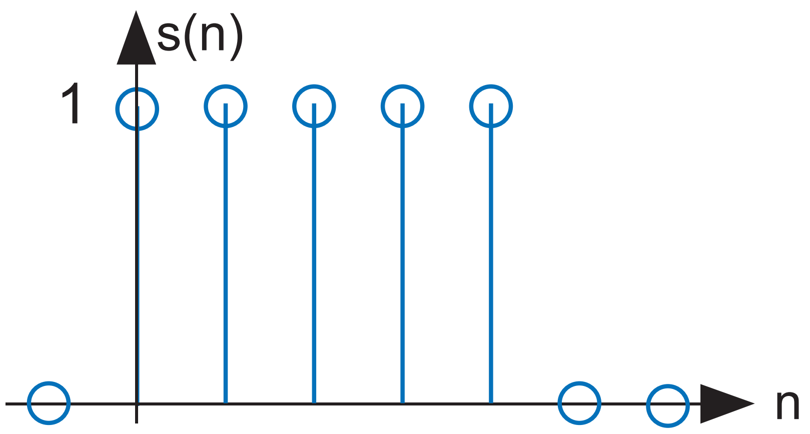
$$\boxed{s(n) \longleftrightarrow S(k)}$$



Note that the IDFT formula produces a *periodic* signal (period K)

Example

$$s(n) = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$



$$S\left(e^{j2\pi f}\right) = \sum_{n=0}^{N-1} 1 \cdot e^{-j2\pi f n}$$ DTFT

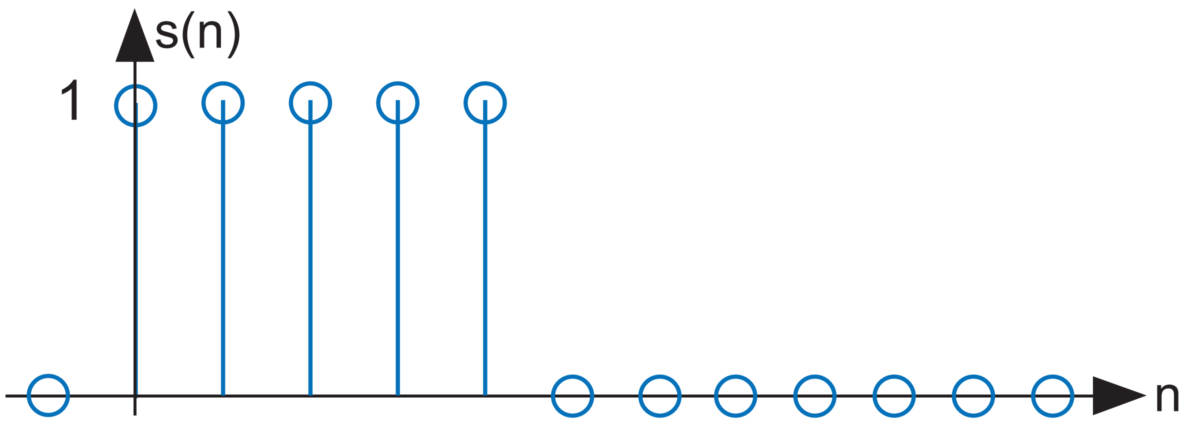
Finite geometric series: $$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$$

$$S\left(e^{j2\pi f}\right) = e^{-j\pi f(N-1)} \frac{\sin \pi f N}{\sin \pi f}$$

$$\text{dsinc}(x) \equiv \frac{\sin Nx}{\sin x}$$

Example

$$s(n) = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$



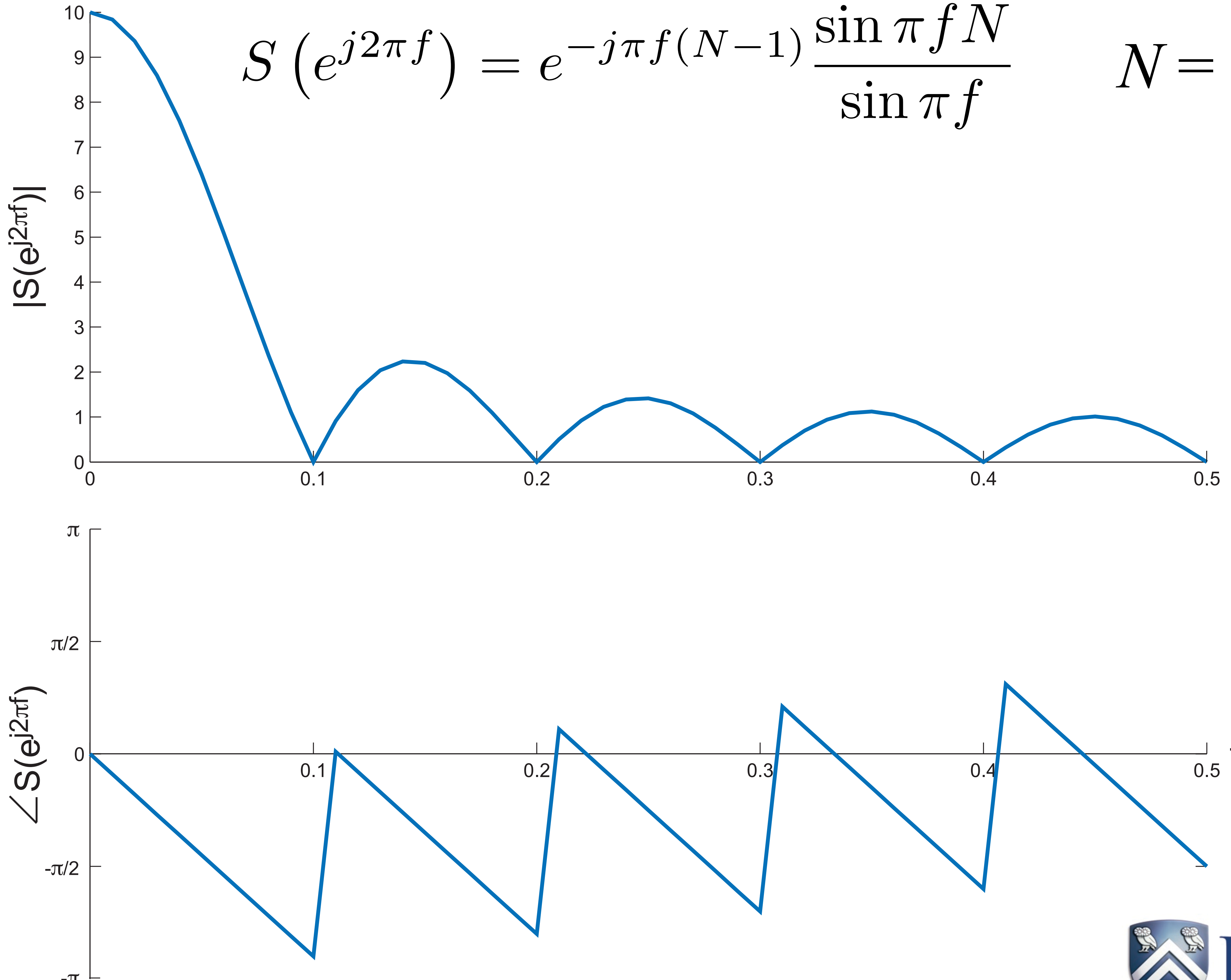
$$S(k) = \sum_{n=0}^{N-1} 1 \cdot e^{-j\frac{2\pi nk}{K}}, \; K \ge N$$

DFT

$$S(k) = e^{-j\frac{\pi k(N-1)}{K}} \frac{\sin \pi Nk/K}{\sin \pi k/K} = e^{-j\pi f(N-1)} \frac{\sin \pi f N}{\sin \pi f} \bigg|_{f=\frac{k}{K}}$$

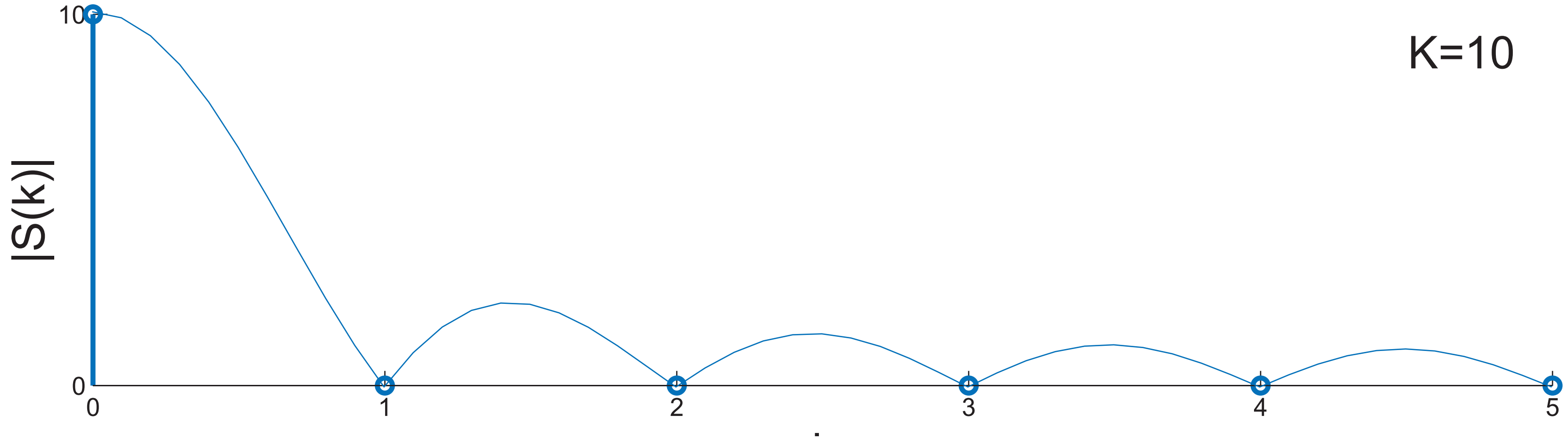
Example

$$S\left(e^{j2\pi f}\right) = e^{-j\pi f(N-1)} \frac{\sin \pi f N}{\sin \pi f} \qquad N = 10$$



Example

$$S(k) = e^{-j\frac{\pi k(N-1)}{K}} \frac{\sin \pi N k/K}{\sin \pi k/K} \qquad N = 10$$



The DFT

- We can use the discrete Fourier transform (DFT) to compute the *sampled* spectrum of *any* discrete-time signal
- Because the DFT is a sampled version of the DTFT, they share similar properties
- Frequently, signals are padded to calculate a transform longer than the signal's duration so as to sample the spectrum more finely
- Remember, positive-frequency values occur for $0 \leq k \leq K/2$, negative-frequency values for $K/2 \leq k \leq K-1$