Fundamentals of Electrical Engineering

The Frequency Domain

- Spectra of non-periodic signals
- Properties of the Fourier transform
- Linear filtering of signals

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Deriving the Fourier Transform

Special notation: $s_T(t)$ periodic with period T

$$c_k(T) = \frac{1}{T}\int_{-T/2}^{T/2} s_T(t) e^{-j\frac{2\pi k t}{T}}\, dt$$

$$S_T(f) \equiv T c_k(T) = \int_{-T/2}^{T/2} s_T(t) e^{-j2\pi f t}\, dt, \;\; f = \frac{k}{T}$$

$$\Longrightarrow \;\; s_T(t) = \sum_{k=-\infty}^{\infty} S_T(f) e^{j2\pi f t} \frac{1}{T}$$

What happens if we extend the period: $T \rightarrow \infty$?

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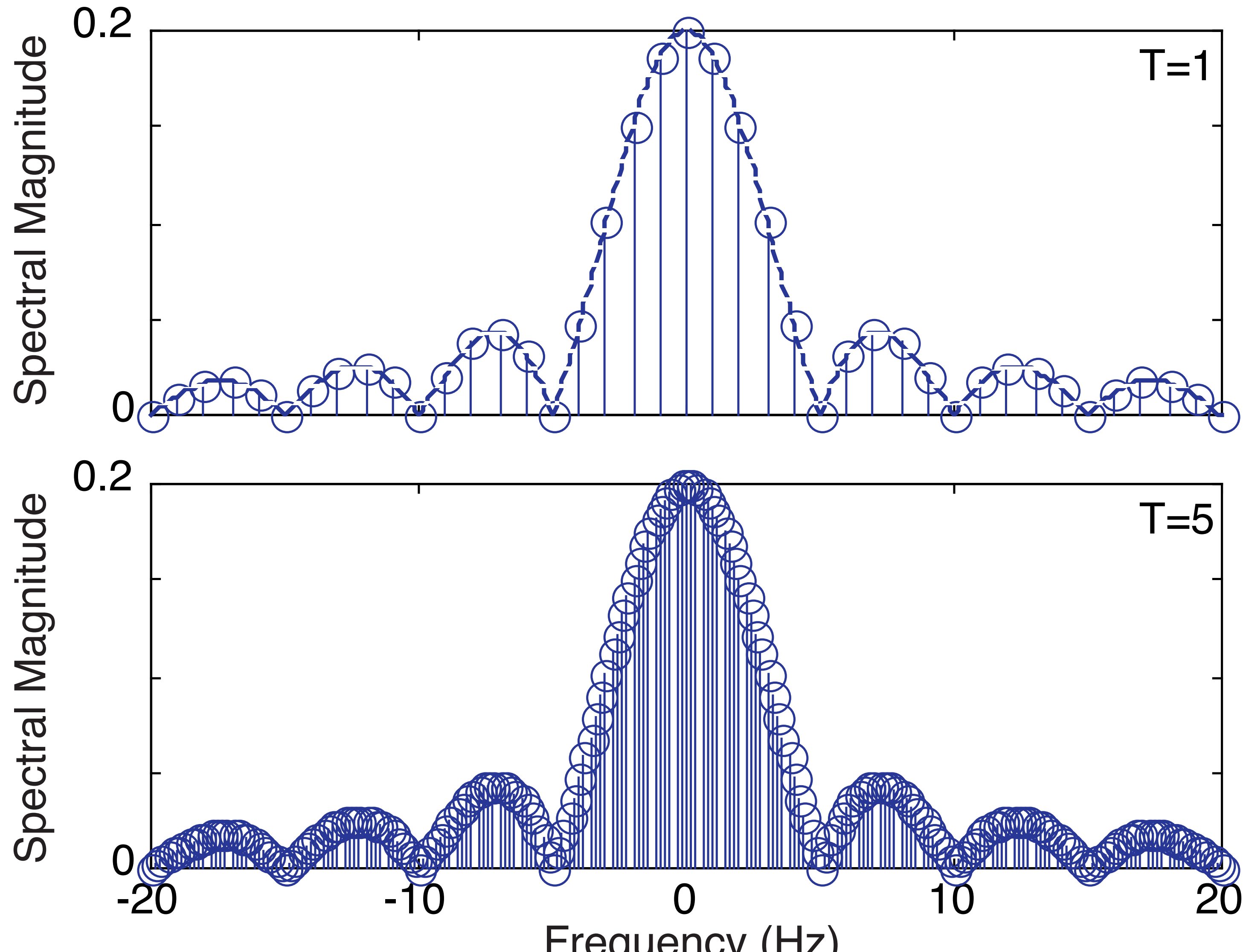
Deriving the Fourier Transform

$$S_T(f) \equiv T c_k(T) = \int_{-T/2}^{T/2} s_T(t) e^{-j2\pi f t}\, dt,\ f = \frac{k}{T}$$

$$s_T(t) = \sum_{k=-\infty}^{\infty} S_T(f) e^{j2\pi f t} \frac{1}{T} \qquad \lim_{T \to \infty} s_T(t) = \int_{-\infty}^{\infty} S(f) e^{j2\pi f t}\, df$$



Fourier Transform Pair

$$s(t) = \int_{-\infty}^{\infty} S(f) e^{j2\pi f t}\, df \qquad S(f) = \int_{-\infty}^{\infty} s(t) e^{-j2\pi f t}\, dt$$

Example: Fourier transform of a pulse

$$\begin{aligned} P(f) &= \int_{-\infty}^{\infty} p(t) e^{-j2\pi f t}\, dt \\ &= \int_{0}^{\Delta} e^{-j2\pi f t}\, dt \\ &= \frac{1}{j2\pi f}\left(1 - e^{-j2\pi f \Delta}\right) \\ &= e^{-j\pi f \Delta} \frac{\sin(\pi f \Delta)}{\pi f} \end{aligned}$$

Important Example

$$s(t) = \int_{-\infty}^{\infty} S(f) e^{j2\pi f t}\, df \qquad S(f) = \int_{-\infty}^{\infty} s(t) e^{-j2\pi f t}\, dt$$

$$e^{-at}\mathrm{u}(t) \longleftrightarrow \frac{1}{a + j2\pi f}$$

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Spectrum Properties

Parseval's Theorem: $$\int_{-\infty}^{\infty} s^2(t)\,dt = \int_{-\infty}^{\infty} |S(f)|^2\,df$$

Symmetries:

$$s(t) \text{ real} \implies S(-f) = S^*(f) \quad \text{(conjugate symmetry)}$$

$$s(t) \text{ even} \implies S(-f) = S(f) \quad (S(f) \text{ real})$$

$$s(t) \text{ odd} \implies S(-f) = -S(f) \quad (S(f) \text{ imaginary})$$

Fourier Transform Properties

$$s(t) = \int_{-\infty}^{\infty} S(f) e^{j2\pi f t}\, df \qquad S(f) = \int_{-\infty}^{\infty} s(t) e^{-j2\pi f t}\, dt$$

$$s(t-\tau) \longleftrightarrow e^{-j2\pi f \tau} S(f)$$

$$e^{+j2\pi f_0 t} s(t) \longleftrightarrow S(f - f_0)$$

Fourier Transform Properties

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$$s(0) = \int_{-\infty}^{\infty} S(f)\, df$$

$$\int_{-\infty}^{\infty} s(t)\, dt = S(0)$$

Fourier Transform Properties

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$$\frac{d\, s(t)}{dt} \longleftrightarrow j2\pi f S(f)$$

$$\int_{-\infty}^{t} s(\alpha)\, d\alpha \longleftrightarrow \frac{1}{j2\pi f} S(f)$$

Fourier Transform Properties

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$$e^{-at}\mathrm{u}(t) \longleftrightarrow \frac{1}{a + j2\pi f} \qquad p(t) \longleftrightarrow e^{-j\pi f \Delta} \frac{\sin(\pi f \Delta)}{\pi f}$$

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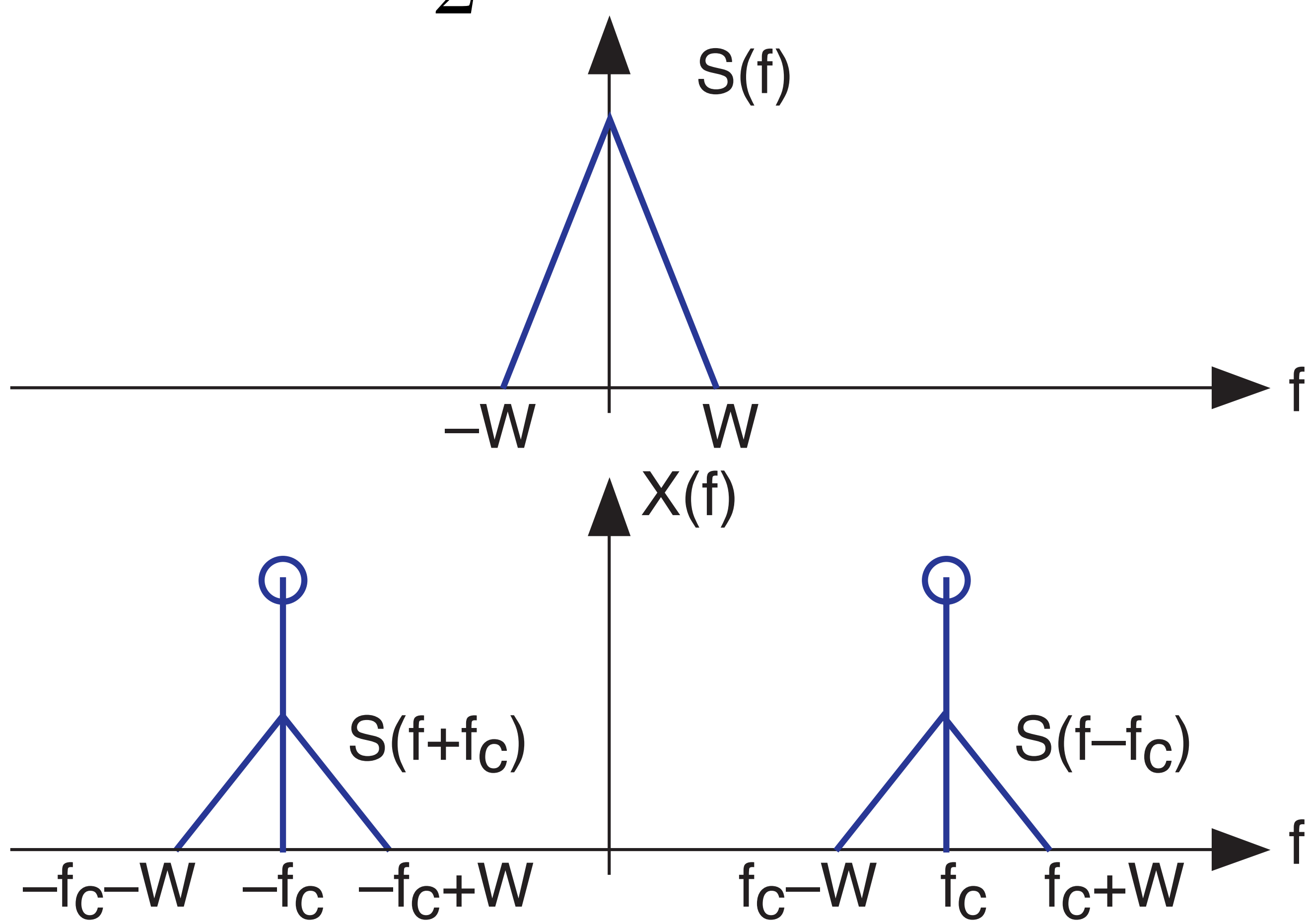
Interesting Example

What is the spectrum of $(1 + s(t))\cos(2\pi f_c t)$?

$$(1 + s(t))\cos(2\pi f_c t) = \cos(2\pi f_c t) + s(t)\cos(2\pi f_c t)$$

$$s(t)\cos(2\pi f_c t) = \frac{1}{2}\left[e^{+j2\pi f_c t} s(t) + e^{-j2\pi f_c t} s(t)\right]$$

$$\leftrightarrow \frac{1}{2}\left[S(f - f_c) + S(f + f_c)\right]$$



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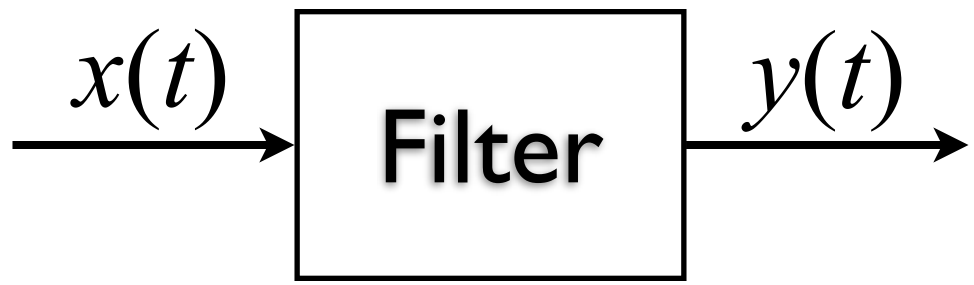
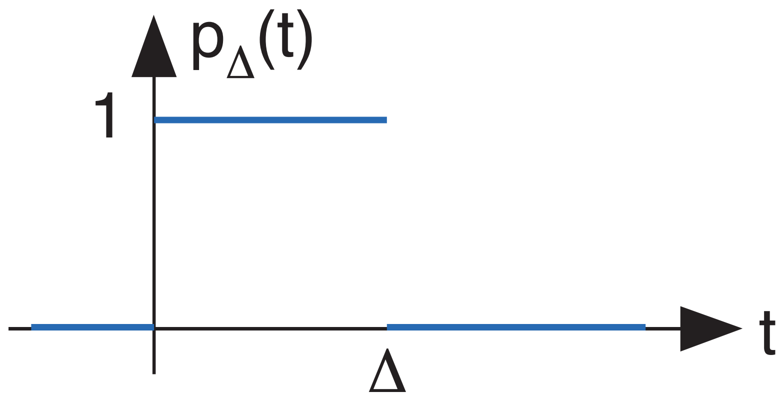
Linear Filters



$$Y(f) = H(f)X(f)$$



$$H(f) = \frac{1}{1 + j2\pi RCf}$$

$$P_\Delta(f) = \frac{1}{j2\pi f}\left(1 - e^{-j2\pi f\Delta}\right)$$

$$Y(f) = \frac{1}{1 + j2\pi f\, RC} \cdot \frac{1}{j2\pi f}\left(1 - e^{-j2\pi f\Delta}\right)$$

Linear Filters

$$Y(f) = \frac{1}{1 + j2\pi f\, RC} \cdot \frac{1}{j2\pi f} \left(1 - e^{-j2\pi f \Delta}\right)$$

$$y(t) = \int_{-\infty}^{\infty} Y(f) e^{+j2\pi f t}\, df$$

$$\underbrace{\frac{1}{j2\pi f}}_{\text{integrate}} \bullet \frac{1}{1 + j2\pi f\, RC} \bullet \underbrace{\left(1 - e^{-j2\pi f \Delta}\right)}_{\text{delay by } \Delta}$$

$$\updownarrow$$

$$\frac{1}{RC} e^{-t/RC} \mathrm{u}(t)$$

$$\left(1 - e^{-t/RC}\right) \mathrm{u}(t) - \left(1 - e^{-(t-\Delta)/RC}\right) \mathrm{u}(t - \Delta)$$

Linear Filters

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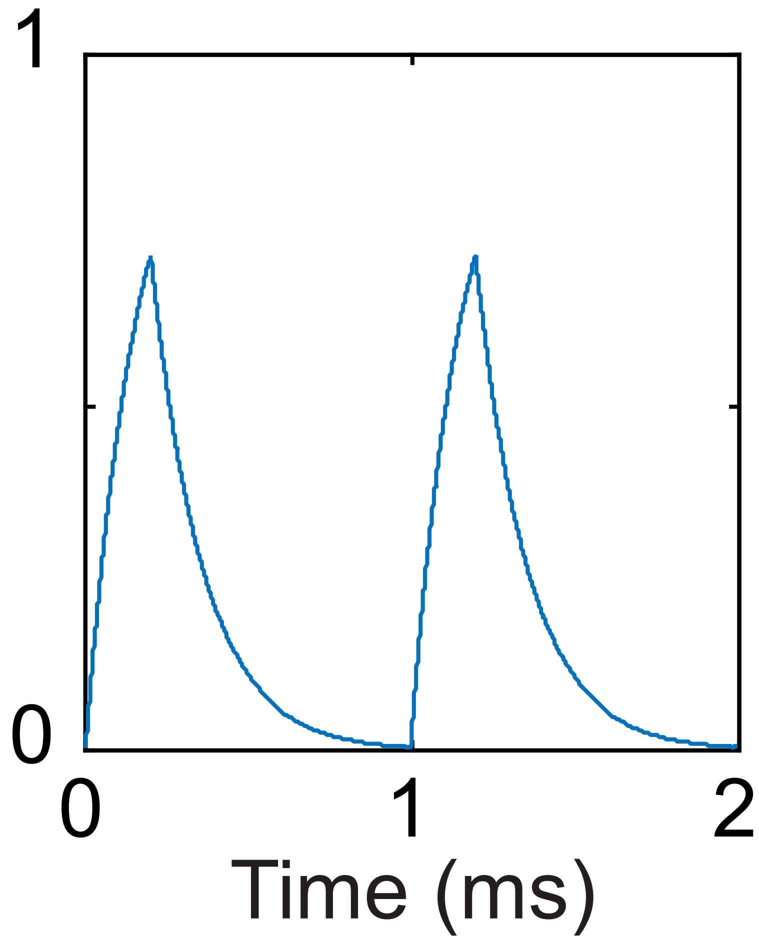

$$\left(1 - e^{-t/RC}\right) \mathrm{u}(t) - \left(1 - e^{-(t-\Delta)/RC}\right) \mathrm{u}(t - \Delta)$$

Signals in the Frequency Domain

- Spectra of periodic signals: Fourier series

$$s(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi k t}{T}} \qquad c_k = \frac{1}{T}\int_0^T s(t) e^{-j\frac{2\pi k t}{T}}\, dt$$

- Spectra on non-periodic signals: Fourier transform

$$s(t) = \int_{-\infty}^{\infty} S(f) e^{j2\pi f t}\, df \qquad S(f) = \int_{-\infty}^{\infty} s(t) e^{-j2\pi f t}\, dt$$

- Linear filters in the frequency domain

$$x(t) \longrightarrow \boxed{\text{Filter}} \longrightarrow y(t)$$

$$y(t) = \sum_{k=-\infty}^{\infty} H\left(\frac{k}{T}\right) c_k e^{j\frac{2\pi k t}{T}} \qquad Y(f) = H(f)X(f)$$