Fundamentals of Electrical Engineering

The Frequency Domain

- Periodic signals: Fourier series
- Signals in time or frequency domains

Fourier Series

$$s(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi k t}{T}} \qquad s(t) \text{ periodic with period } T$$

k^{th} Fourier coefficient

Orthogonality: $$\frac{1}{T}\int_0^T e^{j\frac{2\pi k t}{T}} e^{-j\frac{2\pi l t}{T}}\,dt = \begin{cases} 1 & k = l \\ 0 & k \neq l \end{cases}$$

$$c_l = \frac{1}{T}\int_0^T s(t) e^{-j\frac{2\pi l t}{T}}\,dt$$

$$s(t) \longleftrightarrow c_k$$

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Example: Sinusoid

$$x(t) = A\cos(2\pi f_0 t + \phi) \qquad \text{Period:} \qquad T = \frac{1}{f_0}$$

$$x(t) = \frac{A}{2}\left(e^{-j(2\pi f_0 t + \phi)} + e^{+j(2\pi f_0 t + \phi)}\right)$$

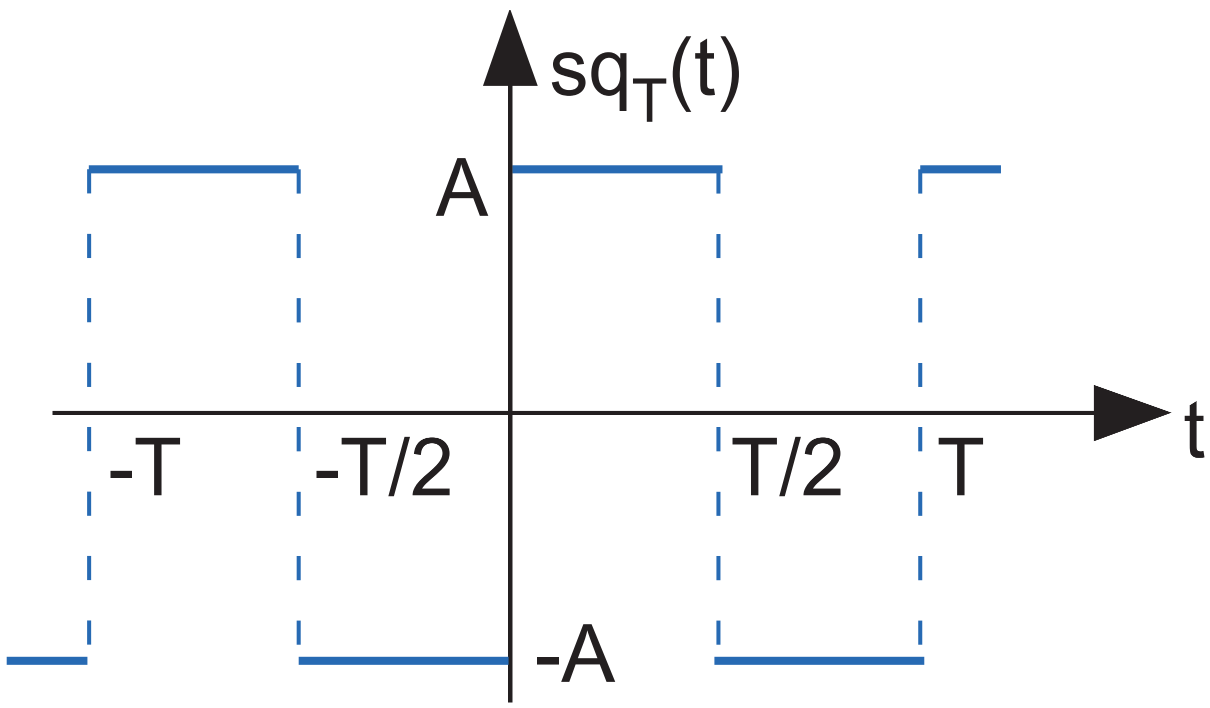
Consider the "symmetric" terms in Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi k t}{T}}$$

$$= \cdots + c_{-k} e^{-j\frac{2\pi k t}{T}} + \cdots + c_k e^{+j\frac{2\pi k t}{T}} + \ldots$$

$$c_{-1} = \frac{A}{2} e^{-j\phi} \quad c_1 = \frac{A}{2} e^{j\phi} \quad c_{\pm k} = 0,\ k \neq 1$$

Example: Square Wave



$$\mathrm{sq}_T(t) = \begin{cases} A & 0 < t < \frac{T}{2} \\ -A & \frac{T}{2} < t < T \end{cases}$$

$$c_k = \frac{1}{T}\int_0^{\frac{T}{2}} A e^{-j\frac{2\pi k t}{T}}\,dt - \frac{1}{T}\int_{\frac{T}{2}}^{T} A e^{-j\frac{2\pi k t}{T}}\,dt$$

Example: Square Wave

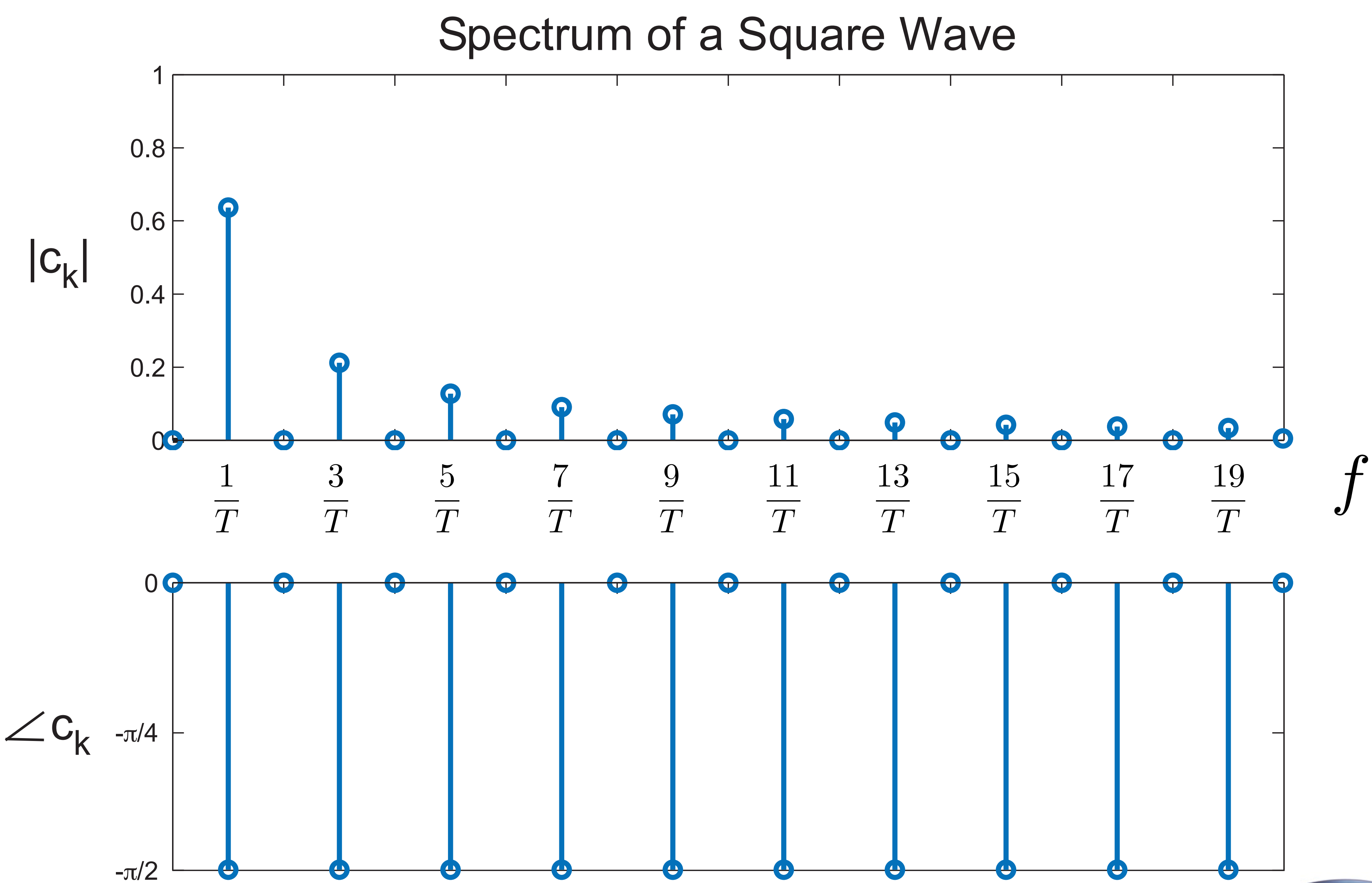
$$c_k = \frac{1}{T}\int_0^{\frac{T}{2}} Ae^{-j\frac{2\pi kt}{T}}\,dt - \frac{1}{T}\int_{\frac{T}{2}}^{T} Ae^{-j\frac{2\pi kt}{T}}\,dt$$

$$c_k = -\frac{2A}{j2\pi k}\left((-1)^k - 1\right)$$

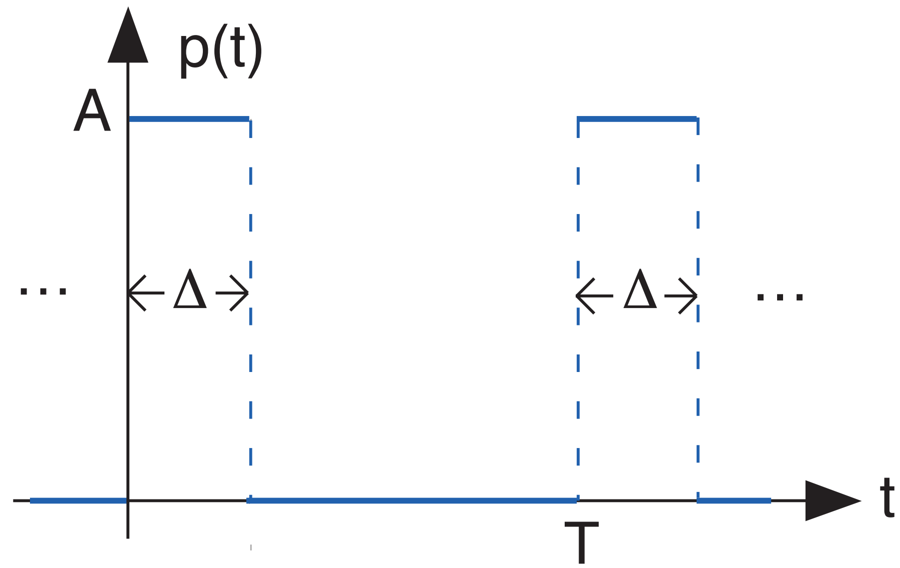
$$= \begin{cases} \frac{2}{j\pi k}A & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

Spectrum of a Square Wave

$$c_k = \begin{cases} \frac{2}{j\pi k} A & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$



Example: Periodic Pulses



$$p(t) = \begin{cases} A, & 0 < t < \Delta \\ 0, & \Delta < t < T \end{cases}$$

$$c_k = \frac{1}{T}\int_0^{\Delta} A e^{-j\frac{2\pi k t}{T}}\,dt = \frac{A}{j2\pi k}\left(1 - e^{-j\frac{2\pi k\Delta}{T}}\right)$$

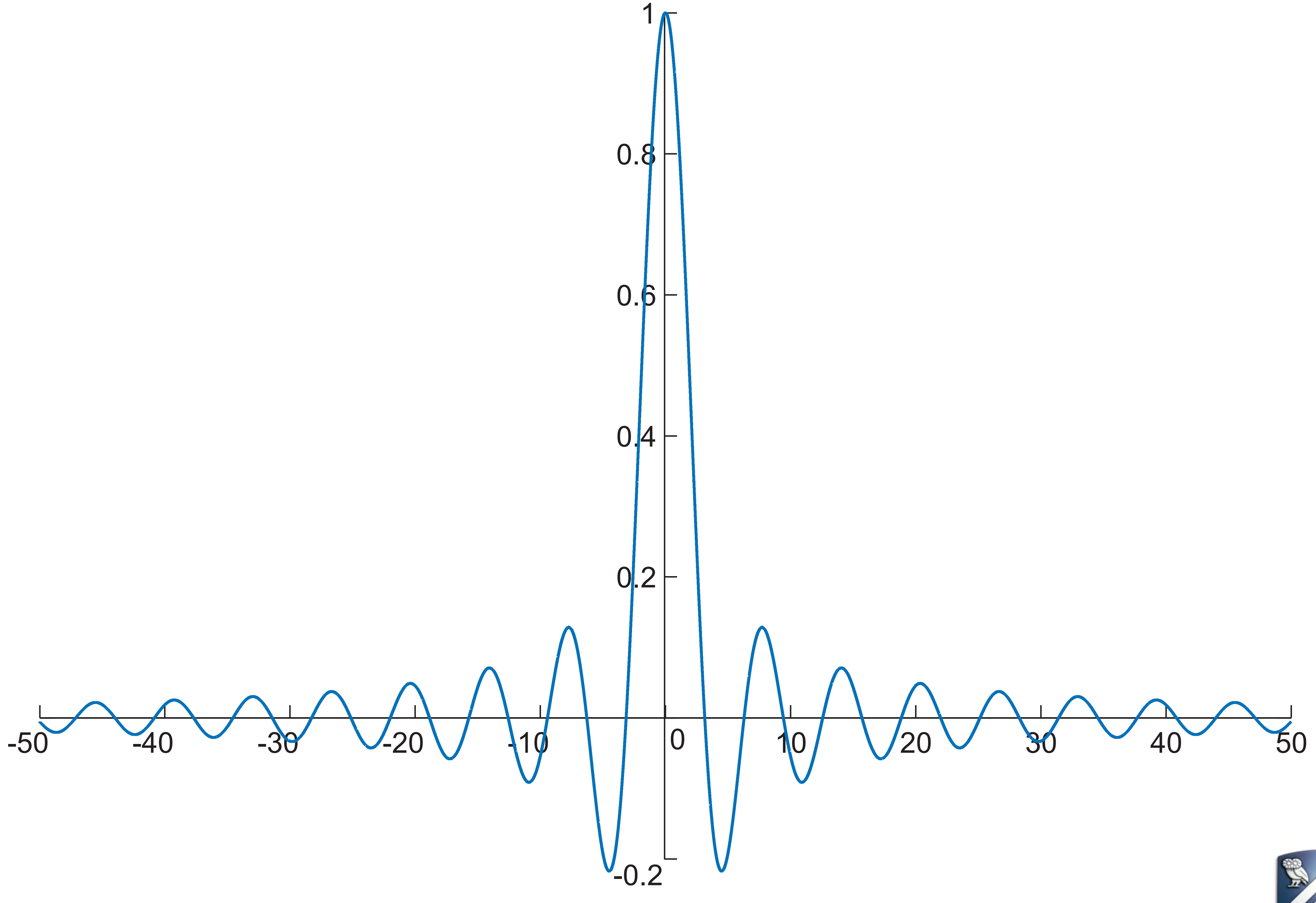
A little math…

$$1 - e^{-j\theta} = e^{-j\frac{\theta}{2}}\left(e^{+j\frac{\theta}{2}} - e^{-j\frac{\theta}{2}}\right) = e^{-j\frac{\theta}{2}}2j\sin\left(\frac{\theta}{2}\right)$$

So $\quad c_k = Ae^{-j\frac{\pi k\Delta}{T}}\frac{\sin\left(\frac{\pi k\Delta}{T}\right)}{\pi k}$

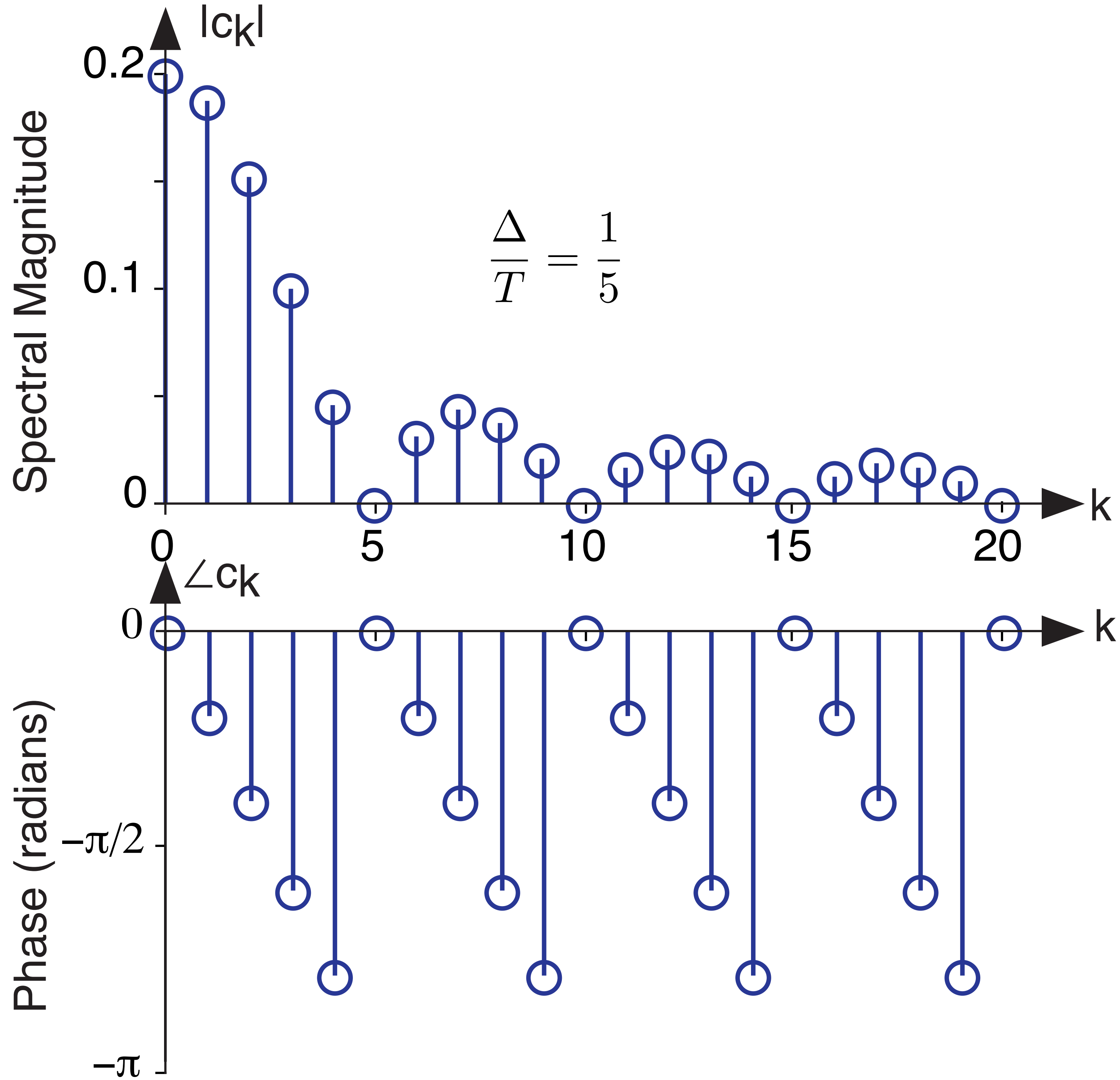
A little (important) math note

$$\mathrm{sinc}(x) \equiv \frac{\sin x}{x}$$

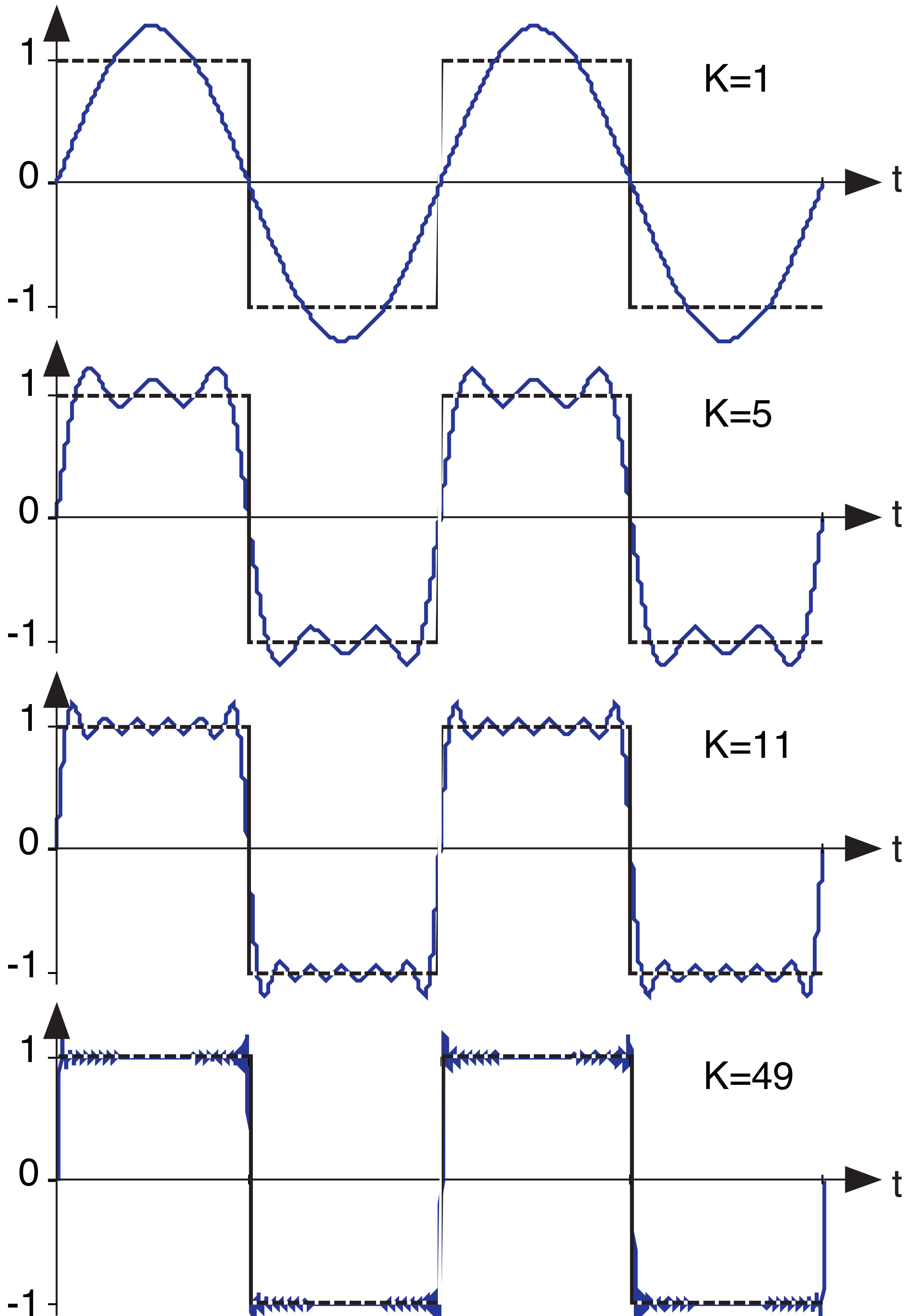


Spectrum of Periodic Pulses

$$c_k = Ae^{-j\frac{\pi k \Delta}{T}} \frac{\sin\left(\frac{\pi k \Delta}{T}\right)}{\pi k} = e^{-j\frac{\pi k \Delta}{T}} \frac{A\Delta}{T} \mathrm{sinc}\left(\frac{\pi k \Delta}{T}\right)$$



Signals in Time and Frequency



$$s(t) \stackrel{?}{=} \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi kt}{T}}$$

$$s(t) = \lim_{K\to\infty} \sum_{k=-K}^{K} c_k e^{j\frac{2\pi kt}{T}}$$

Fourier Series Really Works?

$$s(t) \stackrel{?}{=} \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi k t}{T}}$$

- If $s(t)$ continuous, Fourier series *converges* for each t
- If $s(t)$ has discontinuities, Fourier series does *not* converge at the points of discontinuity
- However,

$$\lim_{K\to\infty} \int_0^T \left[s(t) - \sum_{k=-K}^{K} c_k e^{j\frac{2\pi k t}{T}} \right]^2 dt = 0$$

- Convergence in mean-square (power)

Signals in Time and Frequency

$$s(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi kt}{T}} \qquad c_k = \frac{1}{T}\int_0^T s(t) e^{-j\frac{2\pi kt}{T}}\, dt$$

$$s(t) \longleftrightarrow c_k$$

- If $s(t)$ real-valued, $c_{-k} = c_k^*$
- If $s(t)$ real and even, $s(t) = s(-t)$, $c_{-k} = c_k$ (c_k real and even)
- If $s(t)$ real and odd, $s(t) = -s(-t)$, $c_{-k} = -c_k$ (c_k imaginary)
- $s(t-\tau) \leftrightarrow e^{-j\frac{2\pi k\tau}{T}} c_k$

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Signals in Time and Frequency

$$s(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi kt}{T}} \qquad c_k = \frac{1}{T}\int_0^T s(t) e^{-j\frac{2\pi kt}{T}}\,dt$$

$$s(t) \longleftrightarrow c_k$$

- For periodic signals, the Fourier series represents a way of obtaining the signal's spectrum
- More importantly, signals exist in either the time or frequency domains