Fundamentals of Electrical Engineering

Circuits with Capacitors and Inductors

- Solving circuits for sinusoidal sources

RICE

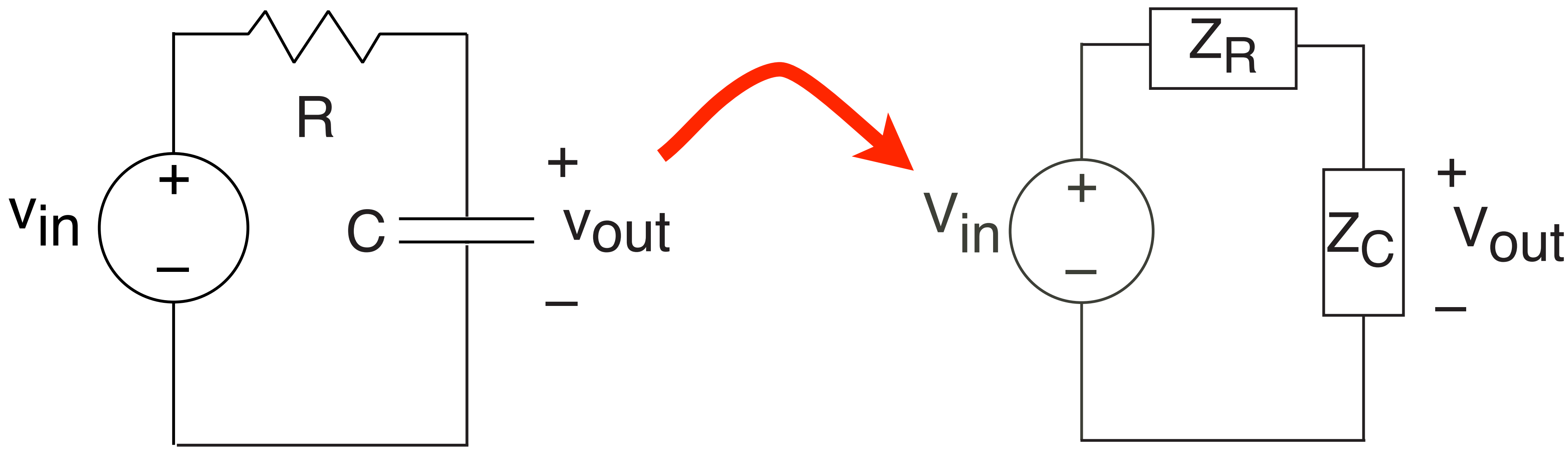
Using Impedances

- The circuit consists of sources and any number of resistors, capacitors and inductors
- Pretend the sources are complex exponentials having a frequency f
- Consider each element an impedance

element	impedance
R	R
C	$\frac{1}{j2\pi fC}$
L	$j2\pi fL$

- Use voltage divider, current divider, series/parallel rules to relate output variable's complex amplitude to the complex amplitude of the source

Sinusoidal Sources



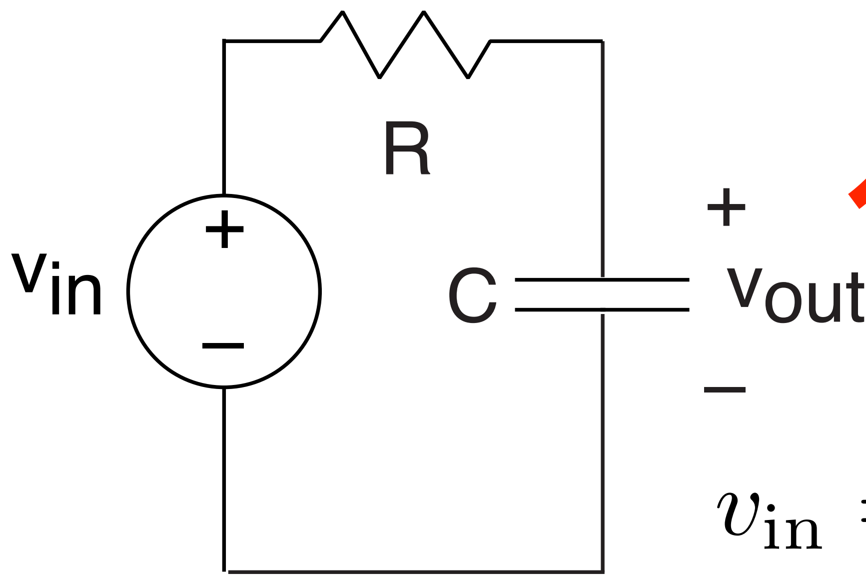
$$V_{\text{out}} e^{j2\pi f t} = \frac{1}{j2\pi f RC + 1} V_{\text{in}} e^{j2\pi f t}$$

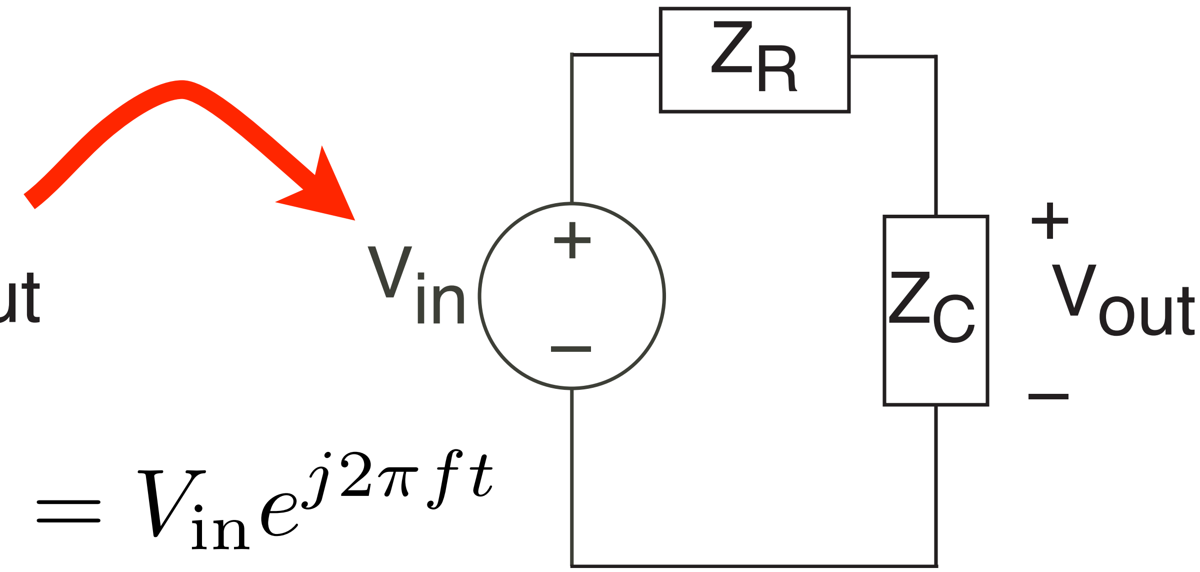
$$v_{\text{in}}(t) = A \cos 2\pi f_0 t \quad = \frac{1}{2}\left(Ae^{j2\pi f_0 t} + Ae^{-j2\pi f_0 t}\right)$$

$$v_{\text{in}}(t) = Ae^{j2\pi f_0 t} \implies v_{\text{out}}(t) = \frac{1}{j2\pi f_0 RC + 1} Ae^{j2\pi f_0 t}$$

$$v_{\text{in}}(t) = Ae^{-j2\pi f_0 t} \implies v_{\text{out}}(t) = \frac{1}{-j2\pi f_0 RC + 1} Ae^{-j2\pi f_0 t}$$

Sinusoidal Sources





$$v_{\text{in}} = V_{\text{in}} e^{j2\pi f t}$$

$$v_{\text{in}}(t) = A\cos 2\pi f_0 t \quad = \frac{1}{2}\left(Ae^{j2\pi f_0 t} + Ae^{-j2\pi f_0 t}\right)$$

Since the circuit elements and KVL/KCL are *linear*, superposition applies

$$v_{\text{out}}(t) = \frac{1}{2}\left(\frac{1}{j2\pi f_0 RC + 1}Ae^{j2\pi f_0 t} + \frac{1}{-j2\pi f_0 RC + 1}Ae^{-j2\pi f_0 t}\right)$$

Sinusoidal Sources

- Note that

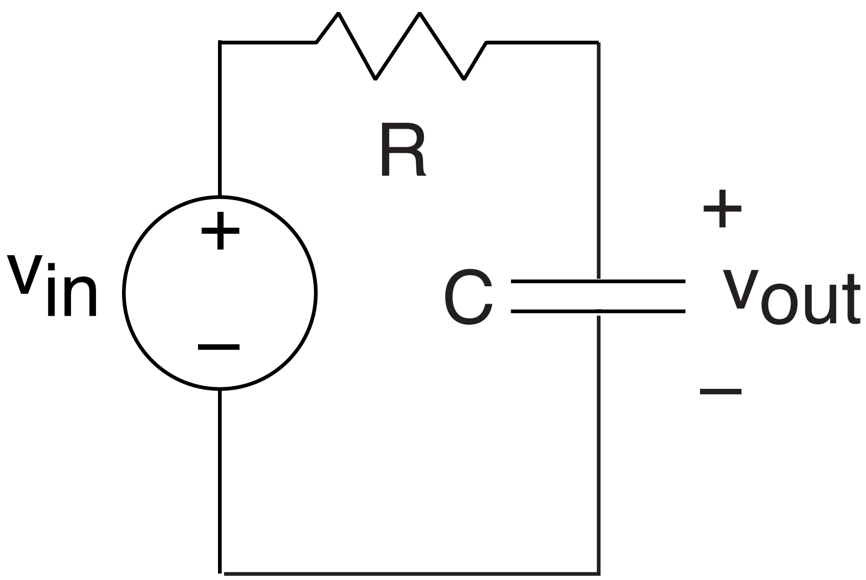
$$v_{\text{in}}(t) = \frac{1}{2}\left(Ae^{j2\pi f_0 t} + Ae^{-j2\pi f_0 t}\right) \qquad v_{\text{in}}(t) = \text{Re}[Ae^{j2\pi f_0 t}]$$

$$v_{\text{out}}(t) = \frac{1}{2}\left(\frac{1}{j2\pi f_0 RC + 1}Ae^{j2\pi f_0 t} + \frac{1}{-j2\pi f_0 RC + 1}Ae^{-j2\pi f_0 t}\right)$$

$$v_{\text{out}}(t) = \text{Re}\left[\frac{1}{j2\pi f_0\, RC + 1}Ae^{j2\pi f_0 t}\right]$$

$$v_{\text{out}}(t) = \text{Re}\left[\frac{1}{\sqrt{4\pi^2 f_0^2 R^2 C^2 + 1}} e^{-j\tan^{-1} 2\pi f_0 RC} Ae^{j2\pi f_0 t}\right]$$

Output for a Sinusoidal Source



$$v_{\text{in}}(t) = A\cos(2\pi f_0 t) = \text{Re}[Ae^{j2\pi f_0 t}]$$

$$V_{\text{out}} = \frac{1}{j2\pi f_0 RC + 1} V_{\text{in}}$$

$$v_{\text{out}}(t) = \text{Re}\left[\frac{1}{j2\pi f_0 RC + 1} Ae^{j2\pi f_0 t}\right]$$

$$v_{\text{out}}(t) = \frac{A}{\sqrt{4\pi^2 f_0^2 R^2 C^2 + 1}} \cos(2\pi f_0 t - \tan^{-1} 2\pi f_0 RC)$$

Sinusoidal Sources

In general, $v_{\text{in}}(t) = A\cos(2\pi f_0 t + \phi) = \text{Re}[\underbrace{Ae^{j\phi}}_{V_{\text{in}}} e^{j2\pi f_0 t}]$

Using impedances, find that $V_{\text{out}} = H(f_0) \cdot V_{\text{in}}$

The *transfer function* $H(f_0) = \dfrac{V_{\text{out}}}{V_{\text{in}}}$ captures the amplitude change and phase shift that the circuit imposes

$$H(f_0) = \frac{1}{j2\pi f_0 RC + 1} = \frac{1}{\sqrt{4\pi^2 f_0^2 R^2 C^2 + 1}} e^{-j\tan^{-1} 2\pi f_0 RC}$$

$$\begin{aligned} v_{\text{out}}(t) &= \text{Re}\left[H(f_0) V_{\text{in}} e^{j2\pi f_0 t}\right] \\ &= |H(f_0)|\,|V_{\text{in}}| \cos\left(2\pi f_0 t + \phi + \angle H(f_0)\right) \end{aligned}$$

RICE

Sinusoidal Sources

Note that if $v_{\text{in}}(t) = A\sin(2\pi f_0 t + \phi) = \text{Im}[Ae^{j\phi}e^{j2\pi f_0 t}]$

$$\begin{aligned} v_{\text{out}}(t) &= \text{Im}\left[H(f_0)V_{\text{in}}e^{j2\pi f_0 t}\right] \\ &= |H(f_0)|\,|V_{\text{in}}|\sin\left(2\pi f_0 t + \phi + \angle H(f_0)\right) \end{aligned}$$

You can use *either* the real or imaginary part in your calculations

$$\cos 2\pi f t = \sin\left(2\pi f t + \frac{\pi}{2}\right) = \text{Im}\left[e^{j\frac{\pi}{2}}e^{j2\pi f t}\right]$$

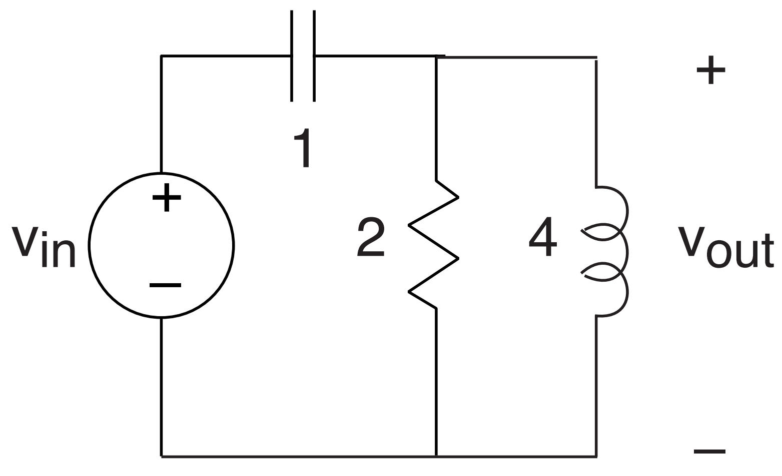
$$\sin 2\pi f t = \cos\left(2\pi f t - \frac{\pi}{2}\right) = \text{Re}\left[e^{-j\frac{\pi}{2}}e^{j2\pi f t}\right]$$

RICE

Using Impedances

- The circuit consists of sources and any number of resistors, capacitors and inductors
- Pretend the sources are complex exponentials having a frequency f
- Consider each element an impedance
- Use voltage divider, current divider, series/parallel rules to relate output variable's complex amplitude to the complex amplitude of the source (the transfer function)
- Express the source as the real (or imaginary) part of a complex exponential
- Output is the real (or imaginary) part of the transfer function times the complex exponential representing the source

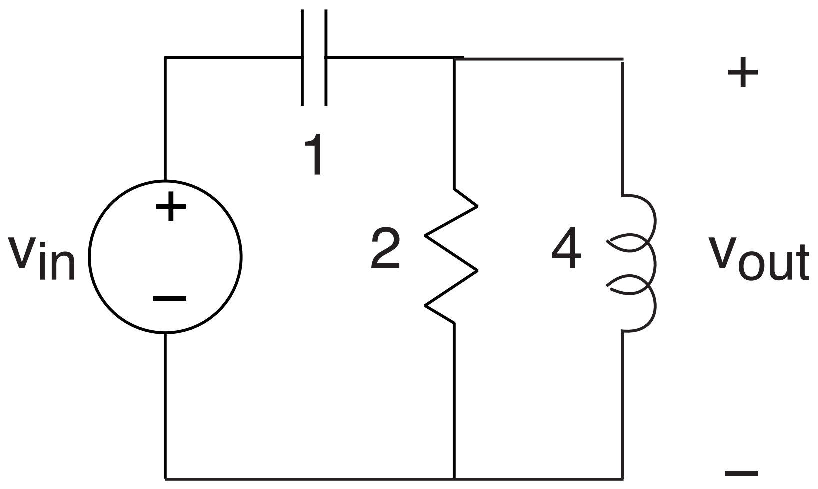
An Example

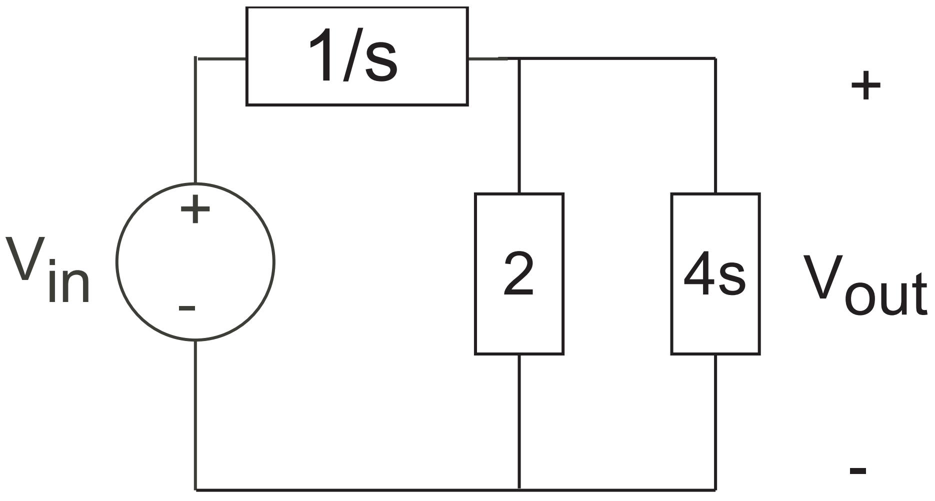


$$v_{\text{in}}(t) = 10\sin(t/2)$$

- What's the transfer function?
- What's the output for the given input voltage?

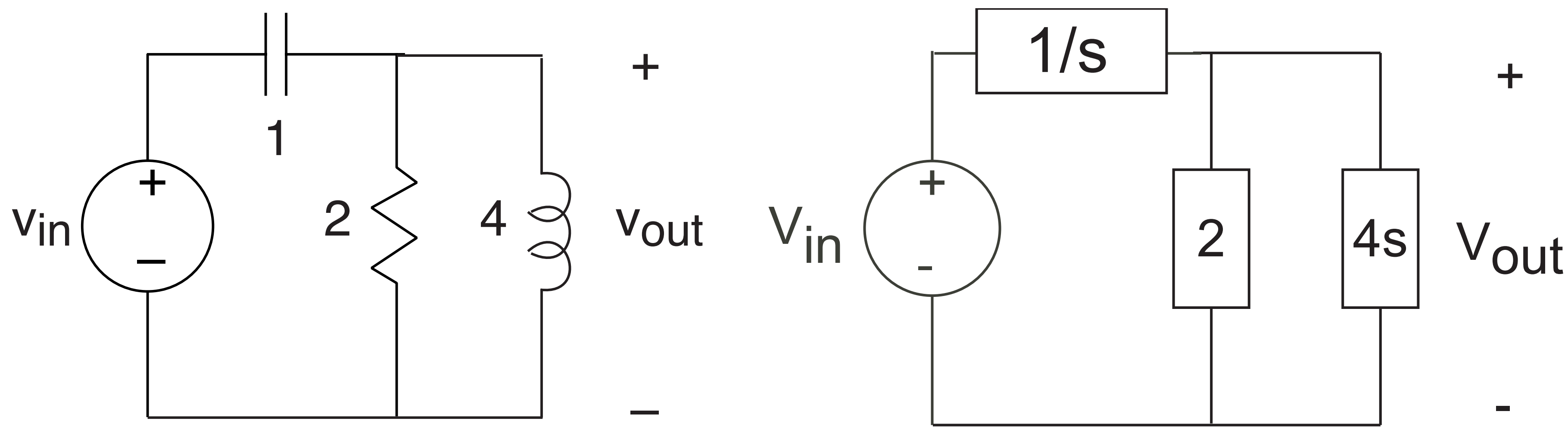
An Example





$$s = j2\pi f$$

An Example



$$v_{\text{in}}(t) = 10\sin(t/2) = \text{Im}\left[10e^{jt/2}\right]$$

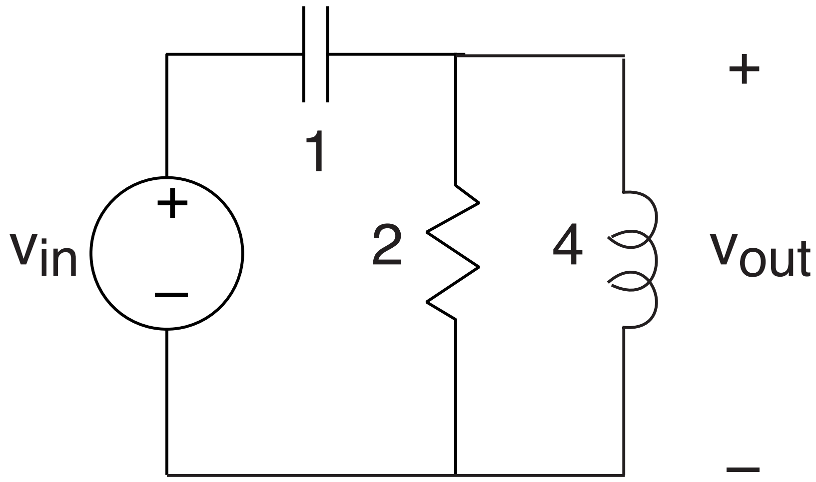
$$H(s) = \frac{4s^2}{4s^2 + 2s + 1} \text{ or } H(f) = \frac{-16\pi^2 f^2}{-16\pi^2 f^2 + j4\pi f + 1}$$

$$2\pi f = \frac{1}{2} \implies f = \frac{1}{4\pi} \qquad H\left(\frac{1}{4\pi}\right) = \frac{-1}{-1 + j + 1} = j$$

$$v_{\text{out}}(t) = \text{Im}[10je^{jt/2}] = \text{Im}\left[10e^{j(t/2+\frac{\pi}{2})}\right] = 10\sin\left(t/2 + \frac{\pi}{2}\right)$$

$$= 10\cos(t/2)$$

What *Not* to Do



$$v_{\text{in}}(t) = 10\sin(t/2) = \text{Im}\left[10e^{jt/2}\right]$$

$$H(f) = \frac{-16\pi^2 f^2}{-16\pi^2 f^2 + j4\pi f + 1}$$

$$v_{\text{out}}(t) = \text{Im}\left[H\left(\frac{1}{4\pi}\right)10e^{jt/2}\right]$$ ✔

~~$$v_{\text{out}}(t) = H\left(\frac{1}{4\pi}\right)10\sin(t/2)$$~~

~~$$v_{\text{out}}(t) = \text{Im}\left[H\left(\frac{1}{4\pi}\right)10\sin(t/2)\right]$$~~