Fundamentals of Electrical Engineering

Complex Numbers

- Definitions
- Complex Arithmetic
- Cartesian and Polar Forms

Complex Numbers

Definition: A complex number z consists of an ordered pair of *real* (ordinary) numbers (a, b) that obeys particular algebraic rules

$$c \cdot z = c \cdot (a, b) = (c \cdot a, c \cdot b), \ c \text{ a scalar}$$

$$z_1 + z_2 = (a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$$

$$z_1 * z_2 = (a_1 a_2 - b_1 b_2, a_1 b_2 + a_2 b_1)$$

$$\frac{1}{z} = \left(\frac{a}{a^2 + b^2}, -\frac{b}{a^2 + b^2} \right)$$

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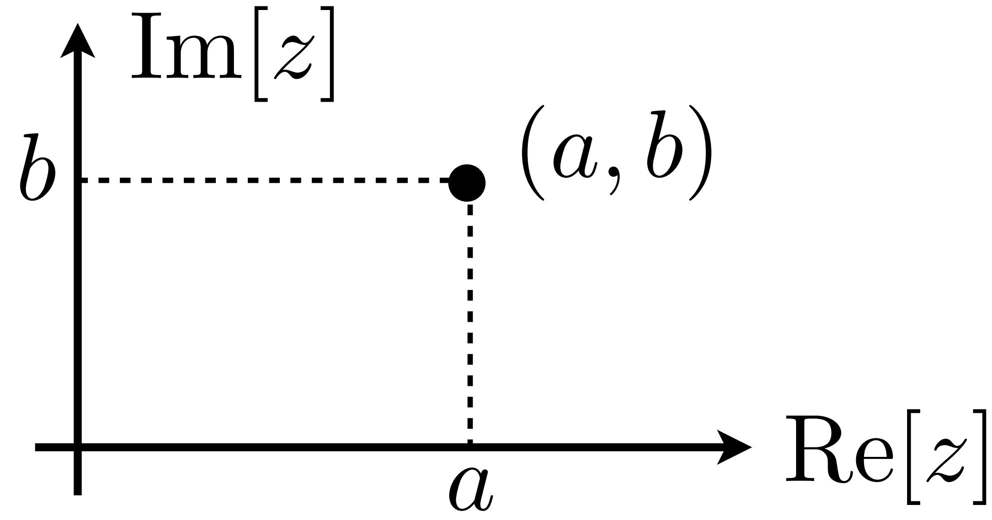
Geometry of Complex Numbers



$$a = \mathrm{Re}[z] \quad b = \mathrm{Im}[z]$$

The algebraic rules can be encapsulated by representing complex numbers by

$$z = a + jb,\ j = \sqrt{-1}$$

and using the "usual" rules

$$c \cdot z = c \cdot (a + jb) = c \cdot a + jc \cdot b$$

$$z_1 + z_2 = (a_1 + jb_1) + (a_2 + jb_2) = (a_1 + a_2) + j(b_1 + b_2)$$

$$\begin{aligned} z_1 \cdot z_2 &= (a_1 + jb_1) \cdot (a_2 + jb_2) \\ &= (a_1 a_2 - b_1 b_2) + j(a_1 b_2 + a_2 b_1) \end{aligned}$$

$$\frac{1}{z} = \frac{a}{a^2 + b^2} - j\frac{b}{a^2 + b^2}$$

Euler's Formula

$$e^{cx} = 1 + cx + \frac{(cx)^2}{2!} + \frac{(cx)^3}{3!} + \frac{(cx)^4}{4!} + \dots$$

Let $c = j$

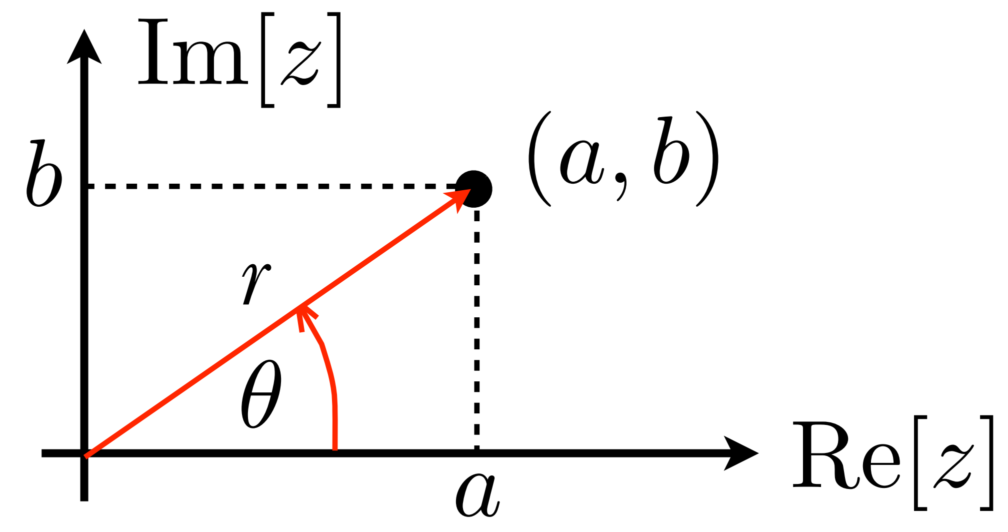
$$e^{jx} = 1 + jx + \frac{(jx)^2}{2!} + \frac{(jx)^3}{3!} + \frac{(jx)^4}{4!} + \dots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + j\left(x - \frac{x^3}{3!}\dots\right)$$

$$= \cos x + j \sin x$$

So $re^{j\theta} = r\cos\theta + jr\sin\theta$ Polar form!!!

More Geometry



$$a = r\cos\theta \quad b = r\sin\theta$$

$$a + jb = r\cos\theta + jr\sin\theta = re^{j\theta}$$

$$r = \sqrt{a^2 + b^2} \quad \theta = \tan^{-1}\frac{b}{a}$$

$z = a + jb$	Cartesian form
$z = re^{j\theta}$	Polar form

Back to those “weird” rules

$$
\begin{aligned}
z_1 \cdot z_2 &= r_1 e^{j\theta_1} \cdot r_2 e^{j\theta_2} \\
&= r_1 r_2 e^{j(\theta_1+\theta_2)} \\
&= r_1 r_2 \cos(\theta_1+\theta_2) + j r_1 r_2 \sin(\theta_1+\theta_2) \\
&= (r_1 \cos\theta_1 \cdot r_2 \cos\theta_2 - r_1 \sin\theta_1 \cdot r_2 \sin\theta_2) \\
&\quad + j\,(r_1 \cos\theta_1 \cdot r_2 \sin\theta_2 + r_1 \sin\theta_1 \cdot r_2 \cos\theta_2) \\
&= (a_1 a_2 - b_1 b_2) + j(a_1 b_2 + a_2 b_1)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{z} &= \frac{1}{re^{j\theta}} = \frac{1}{r} e^{-j\theta} \\
&= \frac{\cos\theta}{r} - j\frac{\sin\theta}{r} \\
&= \frac{a}{a^2+b^2} - j\frac{b}{a^2+b^2}
\end{aligned}
$$

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A Special Operation

The *complex conjugate* of z, denoted by z*, "flips" the sign of the imaginary part

$$z^* = a - jb = re^{-j\theta}$$

So $z \cdot z^* = re^{j\theta} \cdot re^{-j\theta} = r^2$

Magnitude of a complex number: $|z|^2 = z \cdot z^*$ or $|z| = r$

The *real part* of z, denoted by Re[z], equals $\dfrac{z + z^*}{2}$

The *imaginary part* of z, denoted by Im[z], equals $\dfrac{z - z^*}{2j}$

Manipulating Complex Numbers

- Add/subtract in Cartesian form
- Multiply/divide in either form (depends on details)
- Convert between forms before/after operation as needed

$$\frac{1+j}{1+2j} = \frac{(1+j)(1-2j)}{5} = \frac{3-j}{5}$$

Note the multiplication of the numerator and denominator by the conjugate of the denominator.

$$\left|\frac{1+j}{1+2j}\right| = \frac{|1+j|}{|1+2j|} = \frac{\sqrt{2}}{\sqrt{5}} = \sqrt{\frac{2}{5}}$$

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Complex Numbers

- Simplifies calculations
- Very important for signal and system theory