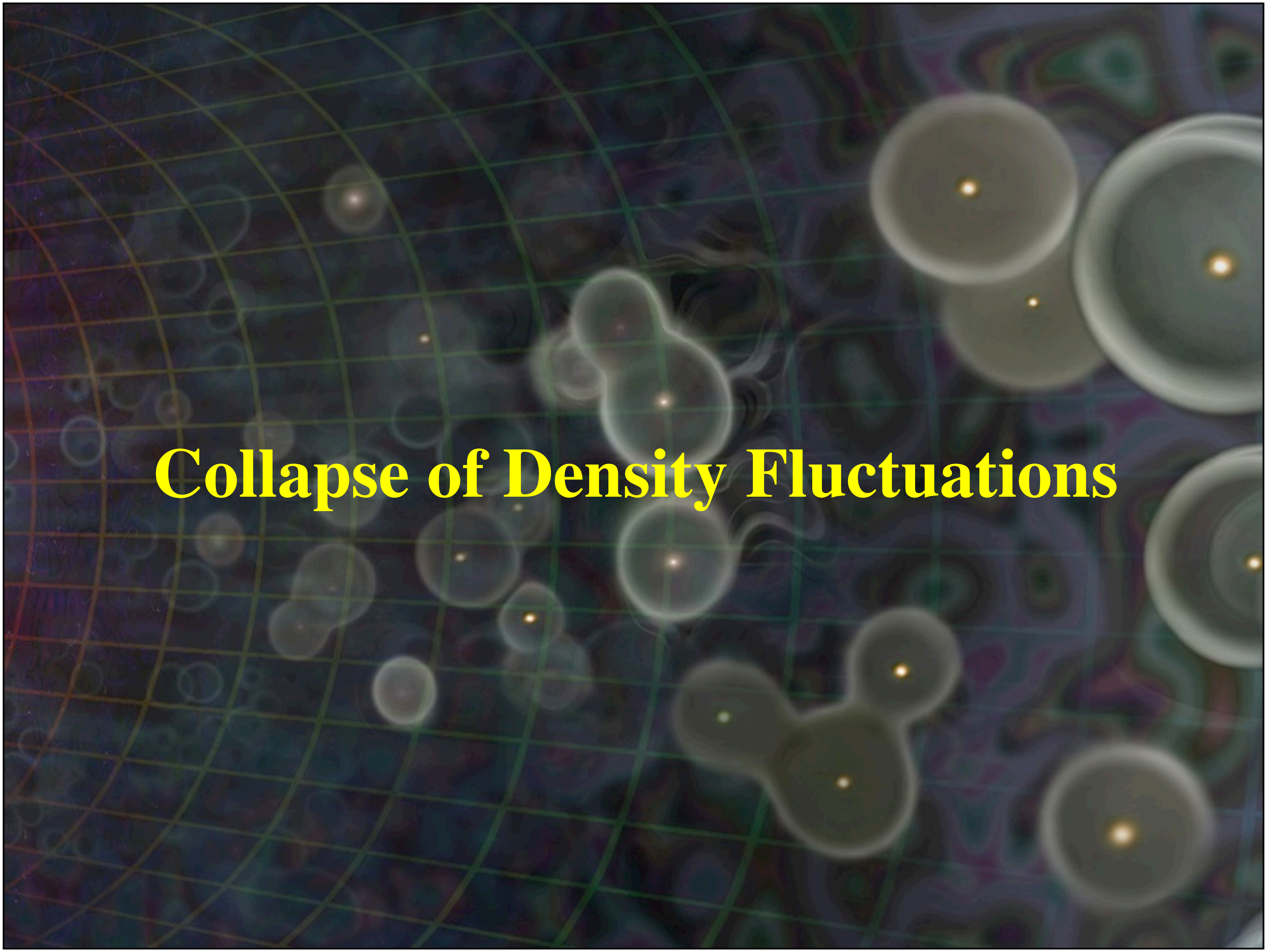
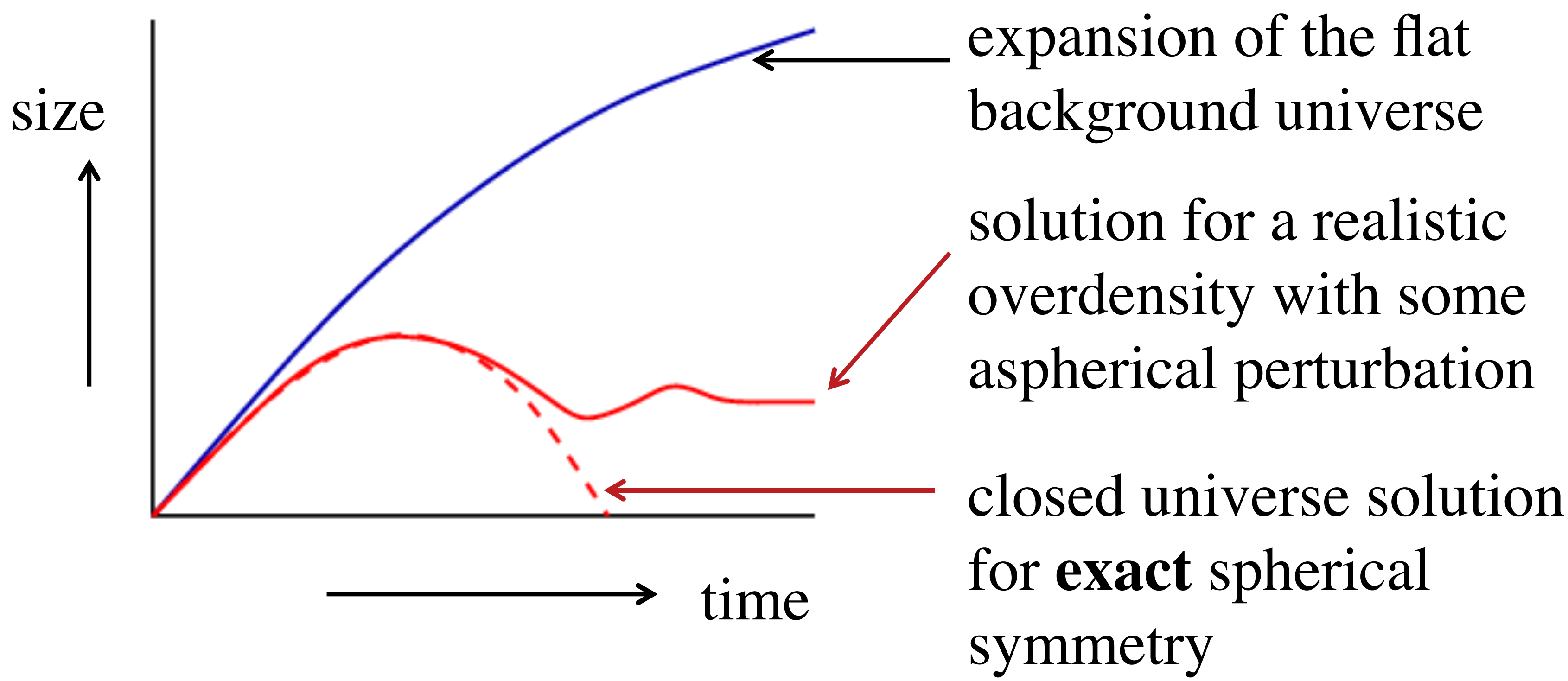
Collapse of Density Fluctuations



Evolution of the Spherical Top-Hat Model



Schematic evolution:

- Density contrast grows as universe expands
- Perturbation “turns around” at $R = R_{turn}$, $t = t_{turn}$
- If exactly spherical, collapses to a point at $t = 2\ t_{turn}$
- Realistically, bounces and **virializes** at radius $R = R_{virial}$

Evolution of the Spherical Top-Hat Model

We can use the virial theorem to derive the final radius of the collapsed perturbation. Let perturbation have mass M, kinetic energy $\langle KE \rangle$, and gravitational potential energy $\langle PE \rangle$:

Virial theorem: $\langle PE \rangle + 2\langle KE \rangle = 0$ (in equilibrium)

Energy conservation: $\langle PE \rangle + \langle KE \rangle = const.$

$$-\frac{GM^2}{R_{turn}} = -\frac{GM^2}{R_{virial}} + \langle KE \rangle = -\frac{GM^2}{2R_{virial}}$$

Energy at turnaround

Energy when virialized

Conclude: $$R_{virial} = \frac{1}{2} R_{turn}$$

Evolution of the Spherical Top-Hat Model

We can apply the solution for a closed universe to calculate the final overdensity:

Friedmann equation: $$\dot{a}^2 = \frac{A^2}{a} - kc^2$$

where $A^2 = \left(\frac{8\pi G}{3}\right)\rho_0 a_0^3$

Solutions:

$$a = \left(\frac{3A}{2}\right)^{2/3} t^{2/3}$$

$k = 0$ solution derived previously

$$a = \frac{1}{2}\frac{A^2}{c^2}(1 - \cos\Psi)$$

$$t = \frac{1}{2}\frac{A^2}{c^3}(\Psi - \sin\Psi)$$

Parametric solution for a closed, $k = 1$ universe (see the derivation elsewhere, e.g., in Ryden's book)

Evolution of the Spherical Top-Hat Model

Now we can calculate the turnaround time for the collapsing sphere by finding when the size of the small closed universe has a maximum:

$$\frac{da}{d\Psi} = \frac{1}{2}\frac{A^2}{c^2}\sin\Psi = 0 \quad \rightarrow \quad \Psi = 0, \pi, \ldots$$

At turnaround, $\Psi = \pi$, which corresponds to time

$$t_{turn} = \frac{1}{2}\frac{A^2}{c^3}\pi$$

The scale factors of the perturbation and of the background universe are:

$$a_{sphere} = \frac{1}{2}\frac{A^2}{c^2}\left(1 - \cos\pi\right) = \frac{A^2}{c^2}$$

$$a_{background} = \left(\frac{3A}{2}\right)^{2/3} t_{turn}^{2/3} = \left(\frac{3\pi}{4}\right)^{2/3}\frac{A^2}{c^2}$$

Evolution of the Spherical Top-Hat Model

The density contrast at turnaround is therefore:

$$\frac{\rho_{sphere}}{\rho_{background}} = \left(\frac{a_{sphere}}{a_{background}}\right)^{-3} = \frac{9\pi^2}{16}$$

At the time when the collapsing sphere virialized, at $t = 2\, t_{turn}$:

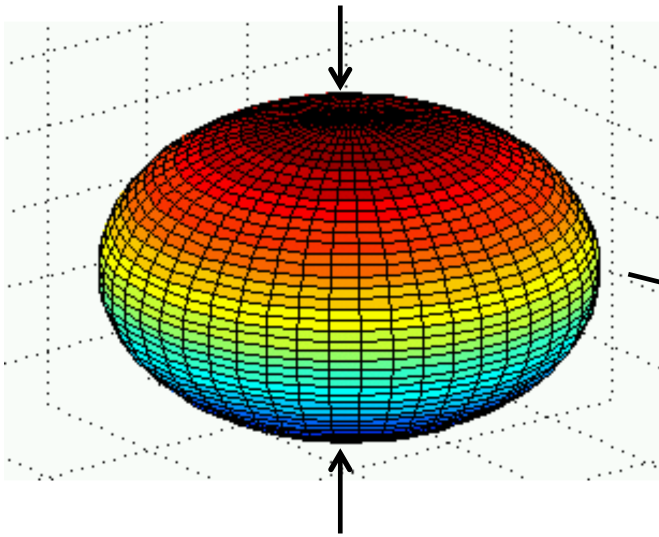
- Its density has increased by a factor of 8 (since $R_{turn} = 2\, R_{vir}$)
- Background density has *decreased* by a factor of $(2^{2/3})^3 = 4$

***A collapsing object virializes when its density is greater than the mean density of the universe by a factor of* $\mathbf{18\,\pi^2 \sim 180}$**
(and this is about right for the large-scale structure today)

First objects to form are small and dense (since they form when the universe is denser). These later merge to form larger structures: **"bottom-up" structure formation**

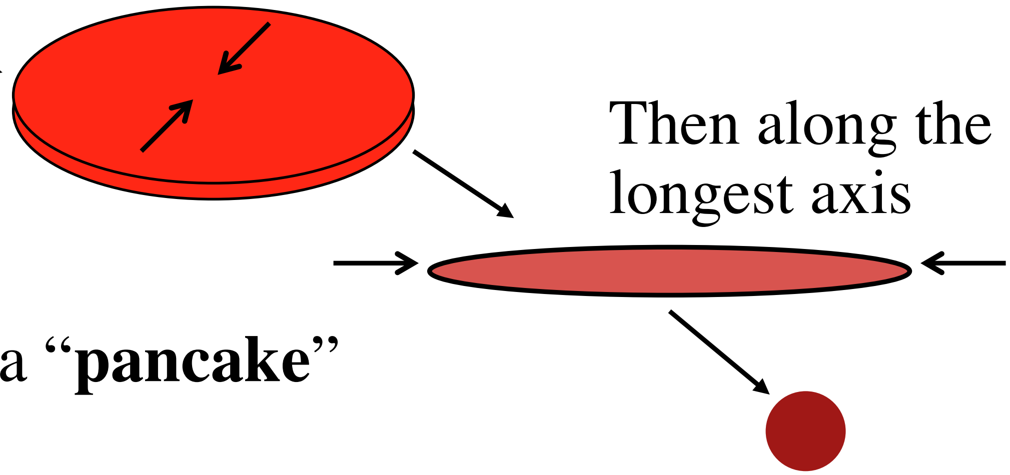
Non-Spherical Collapse

Real perturbations will not be spherical. Consider a collapse of an ellipsoidal overdensity:



The expansion turns into collapse along the shortest axis first …

Then along the intermediate axis



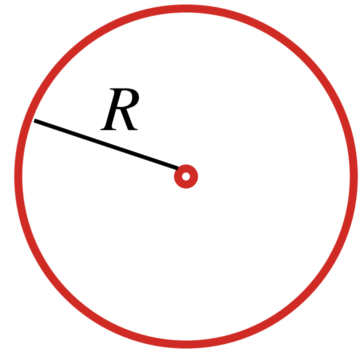
Then along the longest axis

- Perturbation first forms a "**pancake**"
- Then forms **filaments**
- Then forms **clusters**

This kinds of structures are seen both in numerical simulations of structure formation and in galaxy redshift surveys

How Long Does It Take?

The (dissipationless) gravitational collapse timescale is on the order of the free-fall time, t_{ff}:



The outermost shell has acceleration $g = GM/R^2$
It falls to the center in:

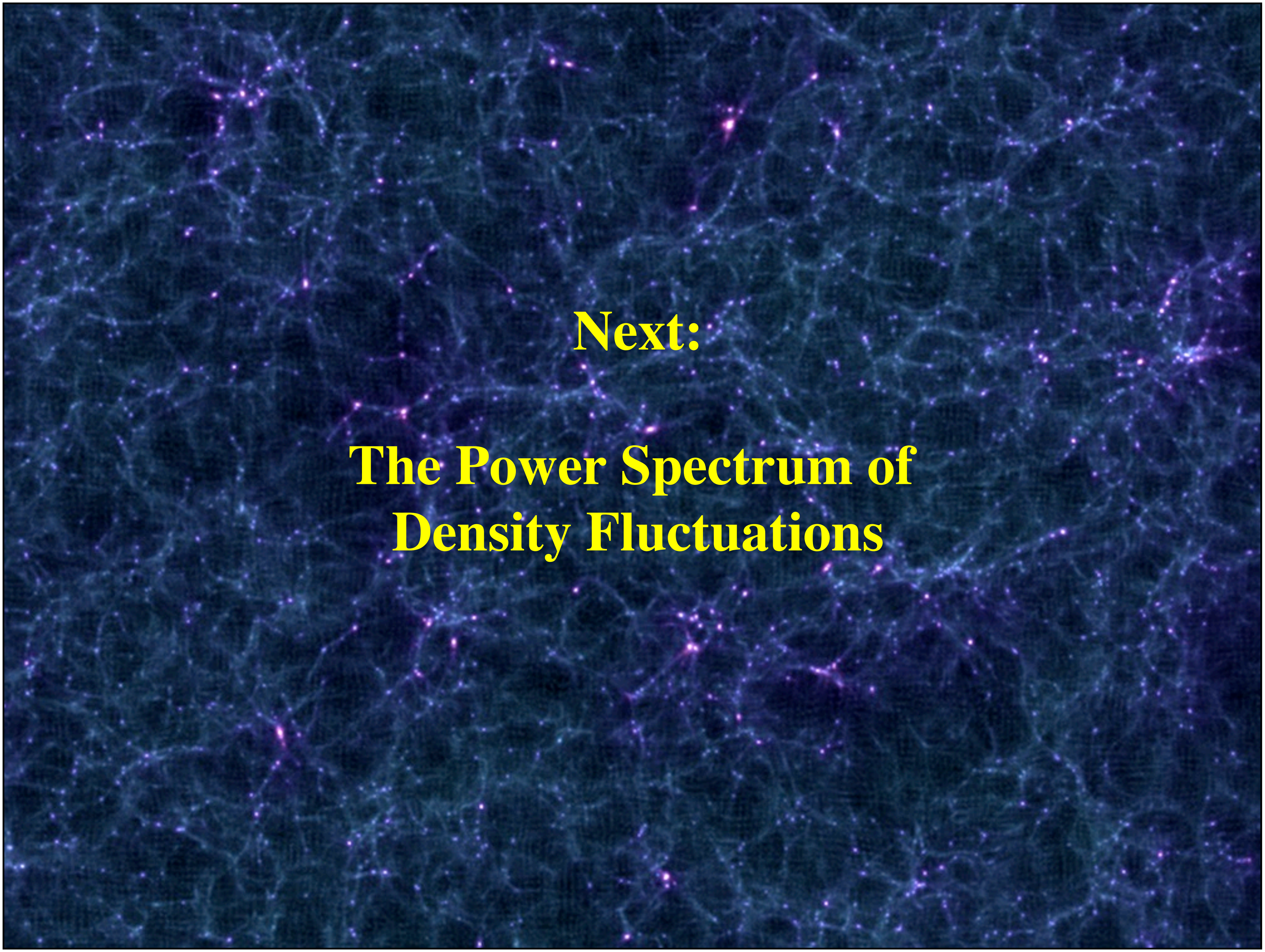
$$t_{ff} = (2R/g)^{1/2} = (2R^3/GM)^{1/2} \approx (2/G\rho)^{1/2}$$

Thus, low density lumps collapse more slowly than high density ones. More massive structures are generally less dense, take longer to collapse. For example:

For a galaxy: $t_{ff} \sim 600\ \mathrm{Myr}\ (R/50\mathrm{kpc})^{3/2}\ (M/10^{12}M_\odot)^{-1/2}$

For a cluster: $t_{ff} \sim 9\ \mathrm{Gyr}\ (R/3\mathrm{Mpc})^{3/2}\ (M/10^{15}M_\odot)^{-1/2}$

So, we expect that galaxies collapsed early (at high redshifts), and that clusters are still forming now. This is as observed!


Next:
The Power Spectrum of
Density Fluctuations