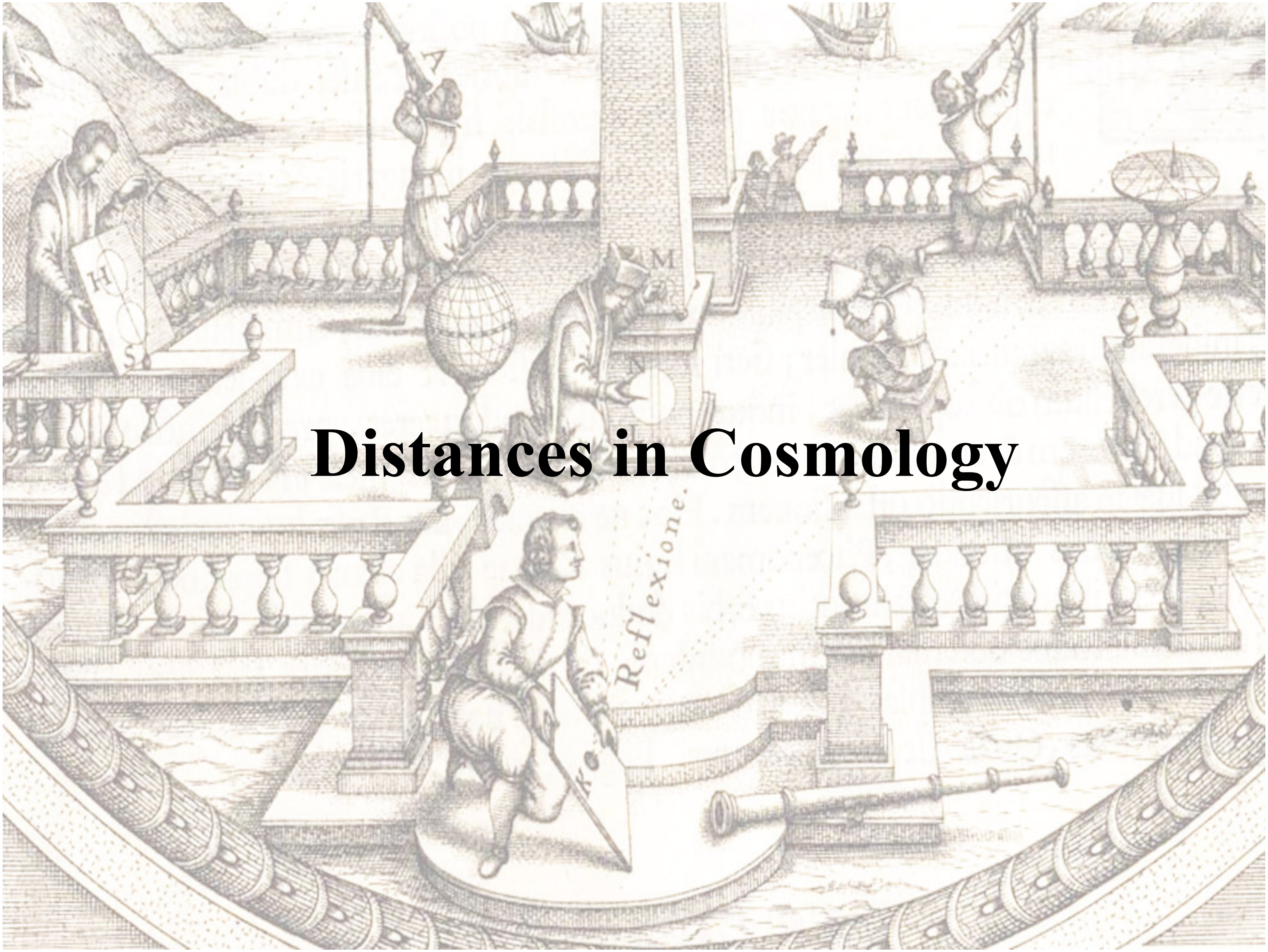
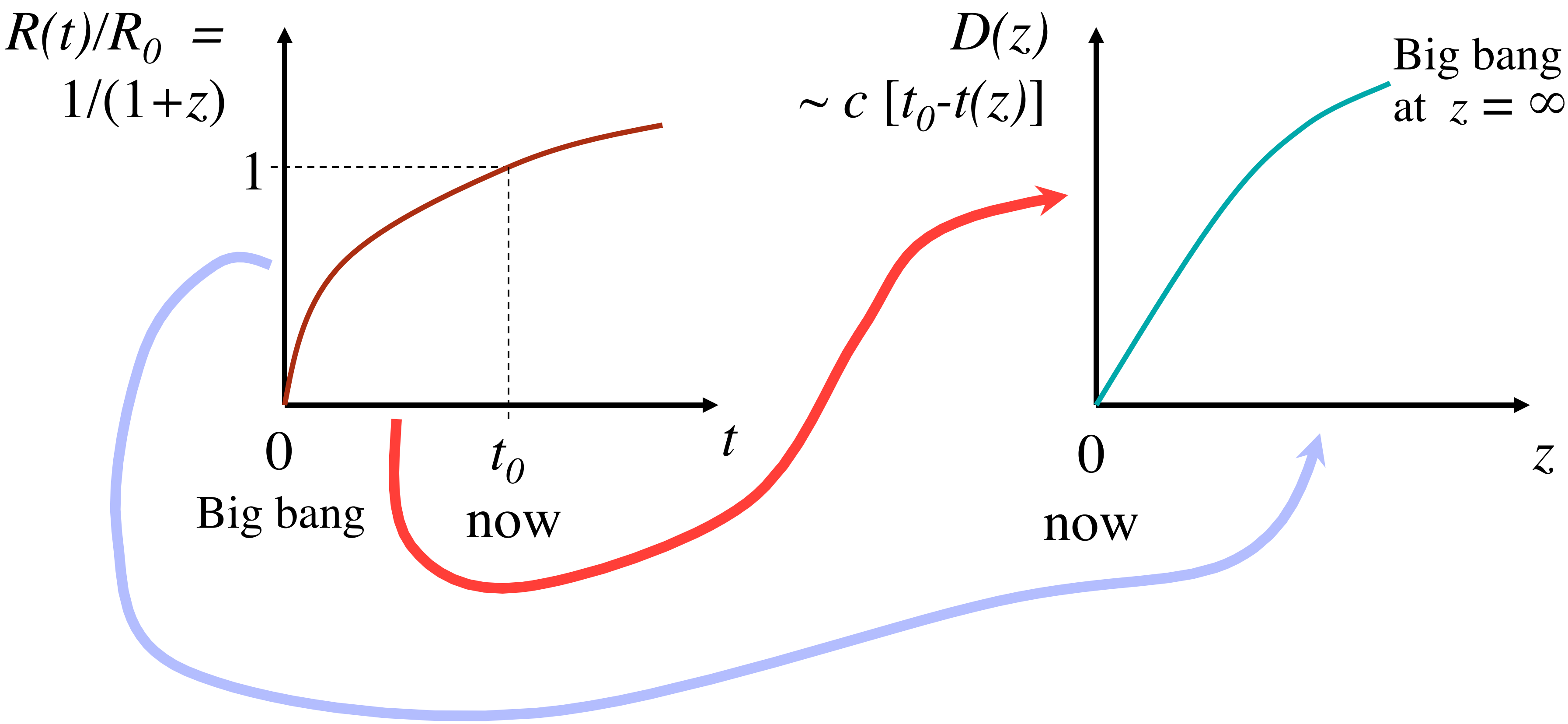
Distances in Cosmology



The Basis of Cosmological Tests



All cosmological tests essentially consist of comparing some measure of (relative) distance (or look-back time) to redshift. Absolute distance scaling is given by the H_0.

Distances in Cosmology

A convenient unit is the **Hubble distance,**

$$D_H = c / H_0 = 4.283\ h_{70}^{-1}\ \text{Gpc} = 1.322 \times 10^{28}\ h_{70}^{-1}\ \text{cm}$$

and the corresponding **Hubble time**,

$$t_H = 1 / H_0 = 13.98\ h_{70}^{-1}\ \text{Gyr} = 4.409 \times 10^{17}\ h_{70}^{-1}\ \text{s}$$

At low z' s, distance $D \approx z\, D_H$. But more generally, the comoving distance to a redshift z is:

$$D_{\rm C} = D_{\rm H} \int_0^z \frac{dz'}{E(z')}$$

where

$$E(z) \equiv \sqrt{\Omega_{\rm M}\,(1+z)^3 + \Omega_k\,(1+z)^2 + \Omega_\Lambda}$$

Distances in Cosmology

But the quantity really useful in computing the various physical quantities of interest is the "transverse comoving distance", where we account for the curvature:

$$D_M = \begin{cases} D_H \frac{1}{\sqrt{\Omega_k}} \sinh\left[\sqrt{\Omega_k}\, D_C/D_H\right] & \text{for } \Omega_k > 0 \\ D_C & \text{for } \Omega_k = 0 \\ D_H \frac{1}{\sqrt{|\Omega_k|}} \sin\left[\sqrt{|\Omega_k|}\, D_C/D_H\right] & \text{for } \Omega_k < 0 \end{cases}$$

where Ω_k is defined by:

$$\Omega_M + \Omega_\Lambda + \Omega_k = 1$$

$$\Omega_M \equiv \frac{8\pi\, G\, \rho_0}{3\, H_0^2}$$

$$\Omega_\Lambda \equiv \frac{\Lambda\, c^2}{3\, H_0^2}$$

Distances in Cosmology

We can derive this for using the RW metric:

$$c^2dt^2 = R^2du^2 = R^2\left\{\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2\right\}$$

To simplify, let's put ourselves at the origin, then the light path is purely radial,
and $d\theta$ and $d\phi = 0$, so:

$$c^2dt^2 = R^2\left\{\frac{dr^2}{1-kr^2}\right\}$$

Taking the square root of both sides and integrating:

$$\int_{t_0}^{t_1} \frac{c}{R}dt = \int_u^0 \frac{dr}{(1-kr^2)^{1/2}}$$

Distances in Cosmology

In general this is non-analytic. In a special case of a $\Lambda = 0$ Universe, we have $q_0 = \Omega_0 / 2$, and:

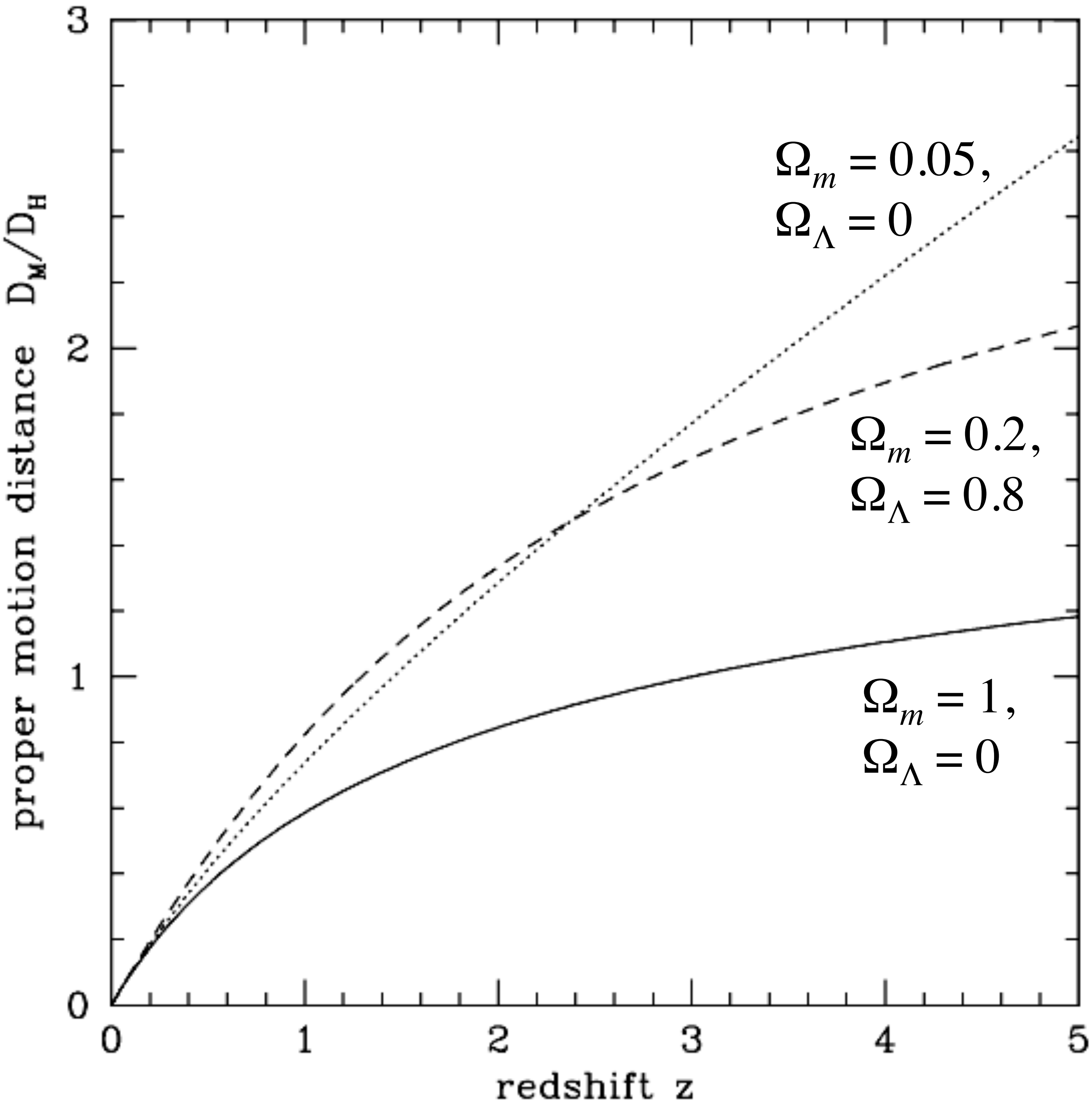
$$d_p = \frac{c}{H_0 q_0^2 (1+z)} \left\{ q_0 z + (q_0 - 1) \left[(2 q_0 z + 1)^{1/2} - 1 \right] \right\}$$

For a non-zero Λ universe:

$$d_p = |\Omega_k|^{-\frac{1}{2}} \sinh \left\{ |\Omega_k|^{\frac{1}{2}} \int_0^z \left\{ (1+z)^2 (1 + \Omega_M z) - \Omega_\Lambda z (2+z) \right\}^{\frac{1}{2}} dz \right\}$$

Assuming $\Omega_k<0$, if $\Omega_k >0$ then the *sinh* becomes a *sin* and if $\Omega_k=0$ then the *sinh* and the Ω_k drop out and all that's left is the integral, which has to be evaluated numerically.

Comoving Distance



Luminosity Distance

In relativistic cosmologies, observed flux (bolometric, or in a finite bandpass) is:

$$f = L \,/\, [\,(4\pi\, D^2)\,(1+z)^2\,]$$

One factor of $(1+z)$ is due to the energy loss of photons, and one is due to the time dialation of the photon rate.

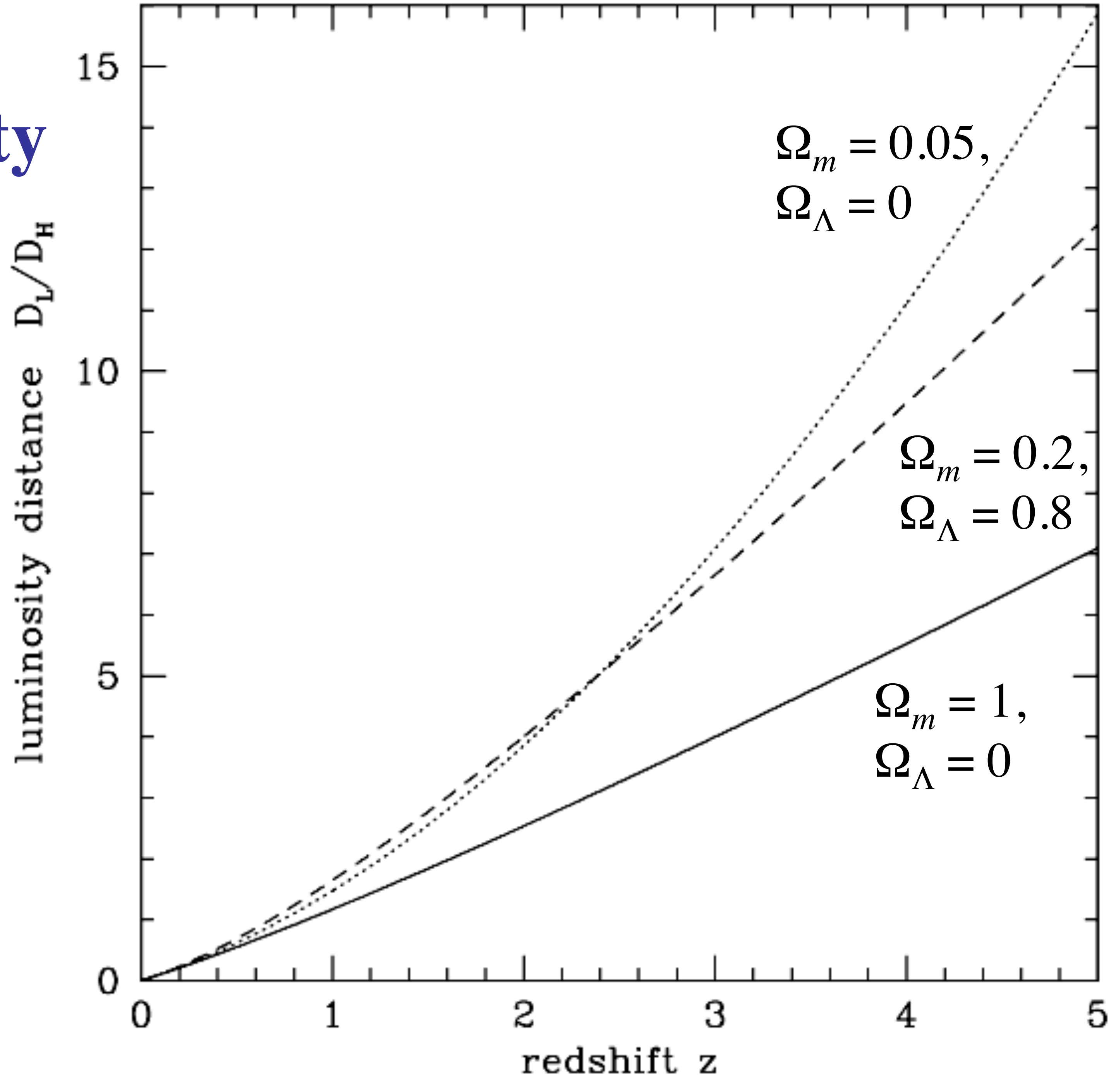
A **luminosity distance** is defined as $D_L = D\,(1+z)$, so that $f = L \,/\, (4\pi\, D_L{}^2)$.

For a specific flux, however,

$$S_\lambda = \frac{1}{(1+z)}\,\frac{L_{\lambda/(1+z)}}{L_\lambda}\,\frac{L_\lambda}{4\pi\, D_{\rm L}^2}$$

(since Angstroms are also stretched by 1+z)

Luminosity Distance



Angular Diameter Distance

Angular diameter of an object with a fixed ***comoving*** size X is by definition

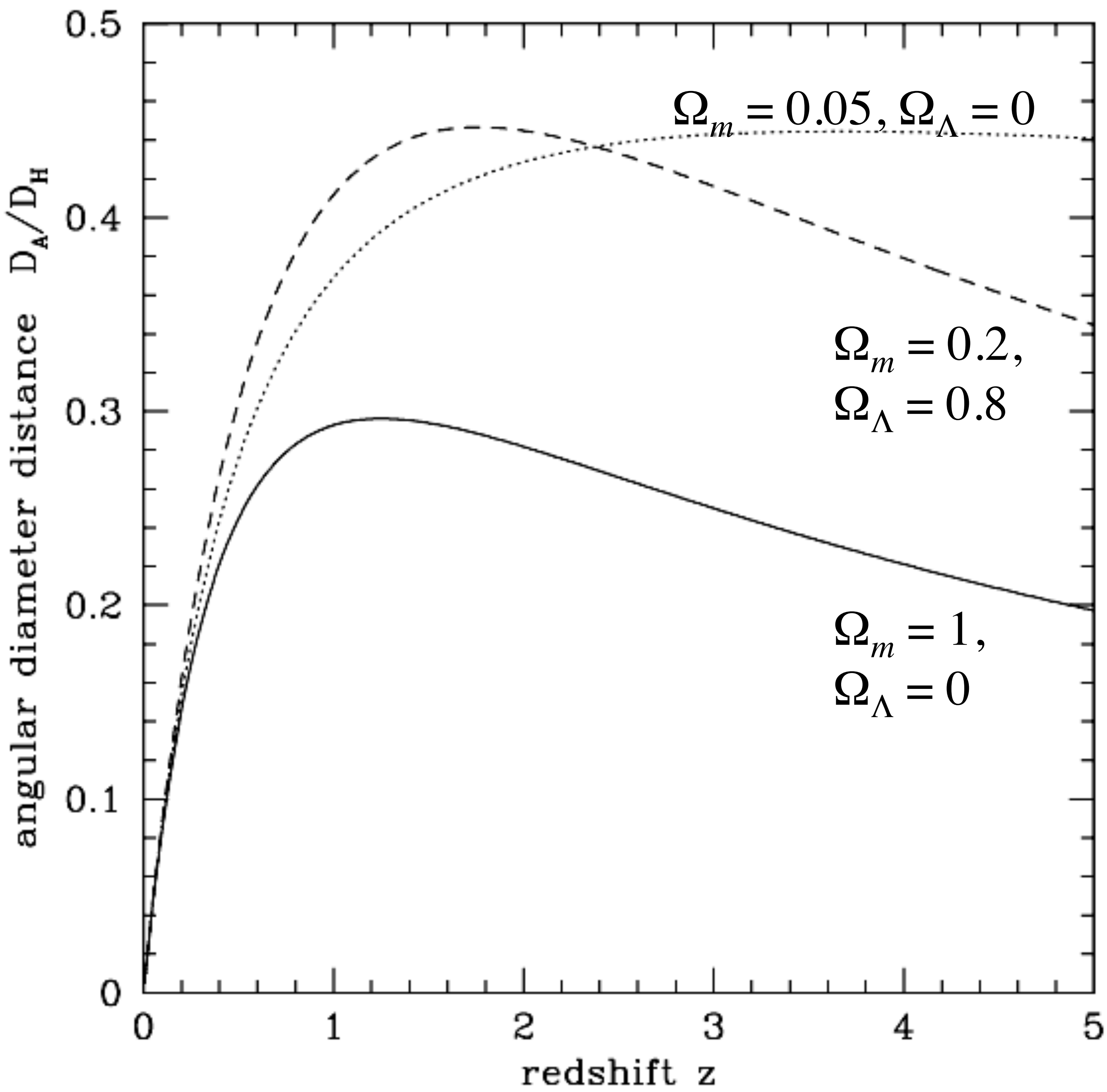
$$\theta = X / D$$

However, an object with a fixed ***proper*** size X is $(1+z)$ times larger than in the comoving coordinates, so its apparent angular diameter will be

$$\theta = (1+z)\ X / D$$

Thus, we define the **angular diameter distance** $D_A = D / (1+z)$, so that the angular diameter of an object whose size is fixed in proper coordinates is $\theta = X / D_A$

Angular Diameter Distance



Volume Element

$$dV_C = D_H \frac{(1+z)^2 D_A^2}{E(z)} \, d\Omega \, dz$$

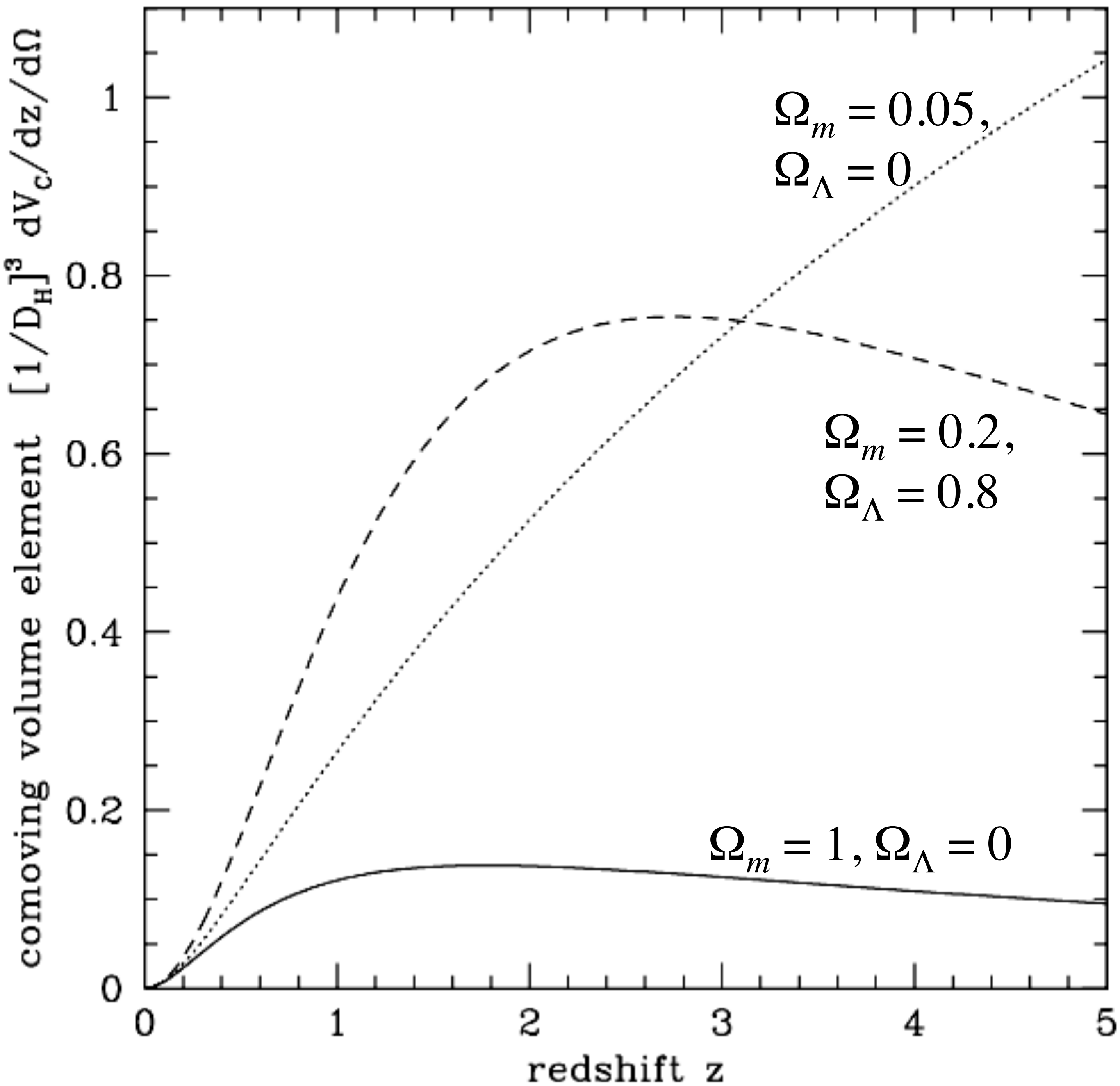
This is useful, e.g., when computing the source counts.

Generally, it has to be evaluated numerically.

The total volume out to some *z*, over the whole sky, is:

$$V_C = \begin{cases} \left(\frac{4\pi D_H^3}{2\Omega_k}\right) \left[\frac{D_M}{D_H}\sqrt{1+\Omega_k \frac{D_M^2}{D_H^2}} - \frac{1}{\sqrt{|\Omega_k|}} \operatorname{arcsinh}\left(\sqrt{|\Omega_k|}\frac{D_M}{D_H}\right)\right] & \text{for } \Omega_k > 0 \\ \frac{4\pi}{3} D_M^3 & \text{for } \Omega_k = 0 \\ \left(\frac{4\pi D_H^3}{2\Omega_k}\right) \left[\frac{D_M}{D_H}\sqrt{1+\Omega_k \frac{D_M^2}{D_H^2}} - \frac{1}{\sqrt{|\Omega_k|}} \arcsin\left(\sqrt{|\Omega_k|}\frac{D_M}{D_H}\right)\right] & \text{for } \Omega_k < 0 \end{cases}$$

Volume Element



Age and Lookback Time

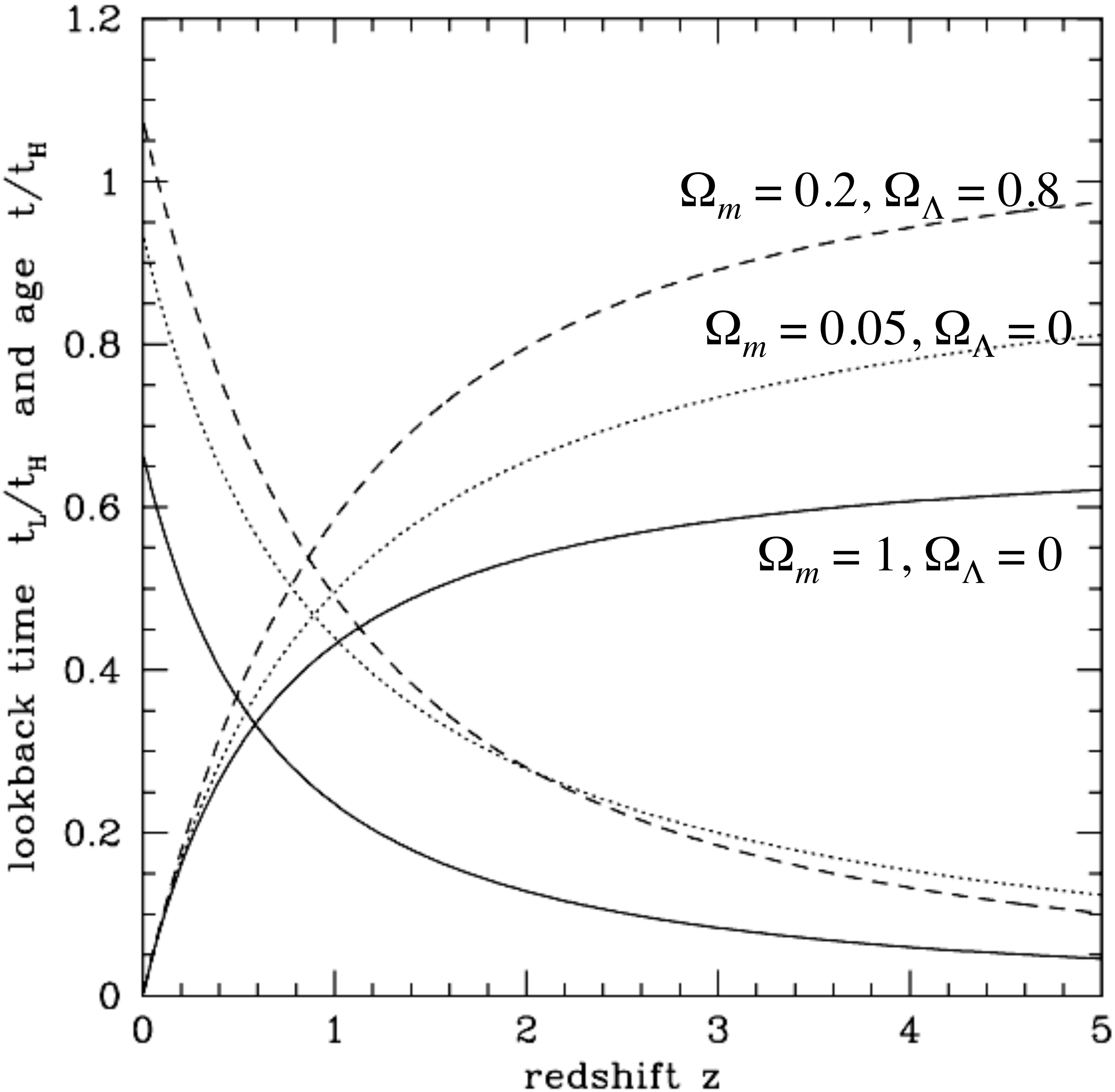
The time elapsed since some redshift z is:

$$t_{\rm L} = t_{\rm H} \int_0^z \frac{dz'}{(1+z')\,E(z')}$$

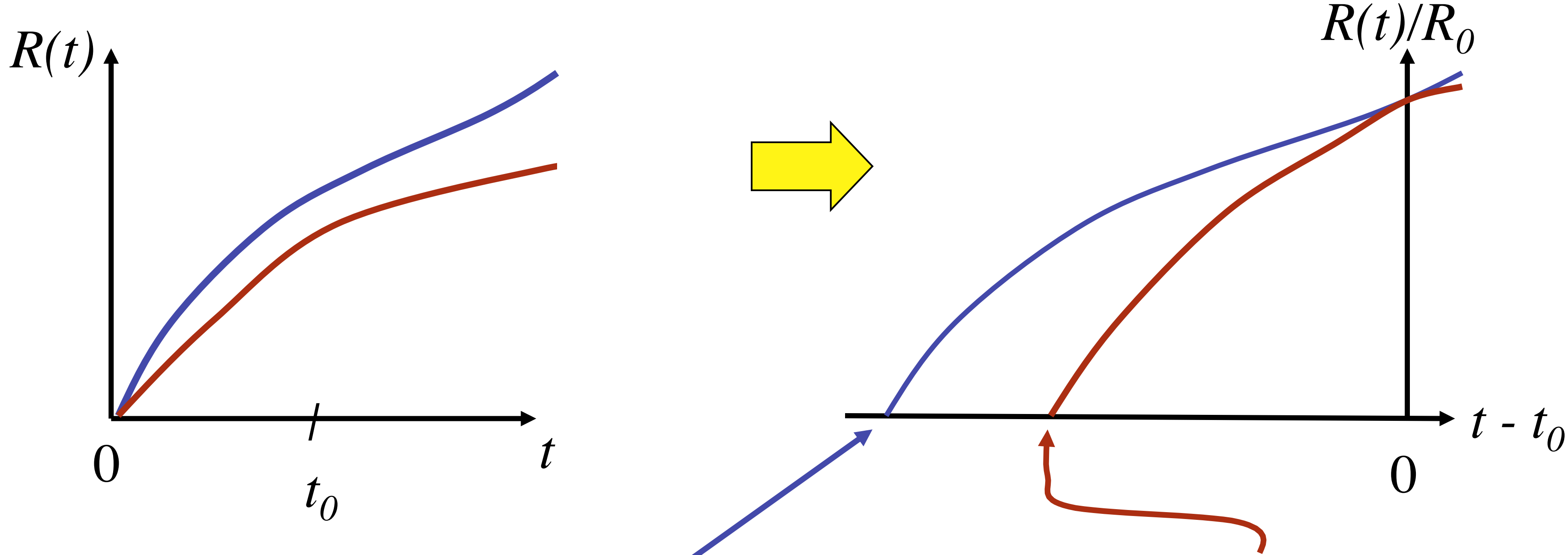
Generally it has to be integrated numerically, except in some special cases, such as $\Lambda = 0$.

Integrating to infinity gives the age of the universe, and the difference of the two is the age at a given redshift.

Age and Lookback Time

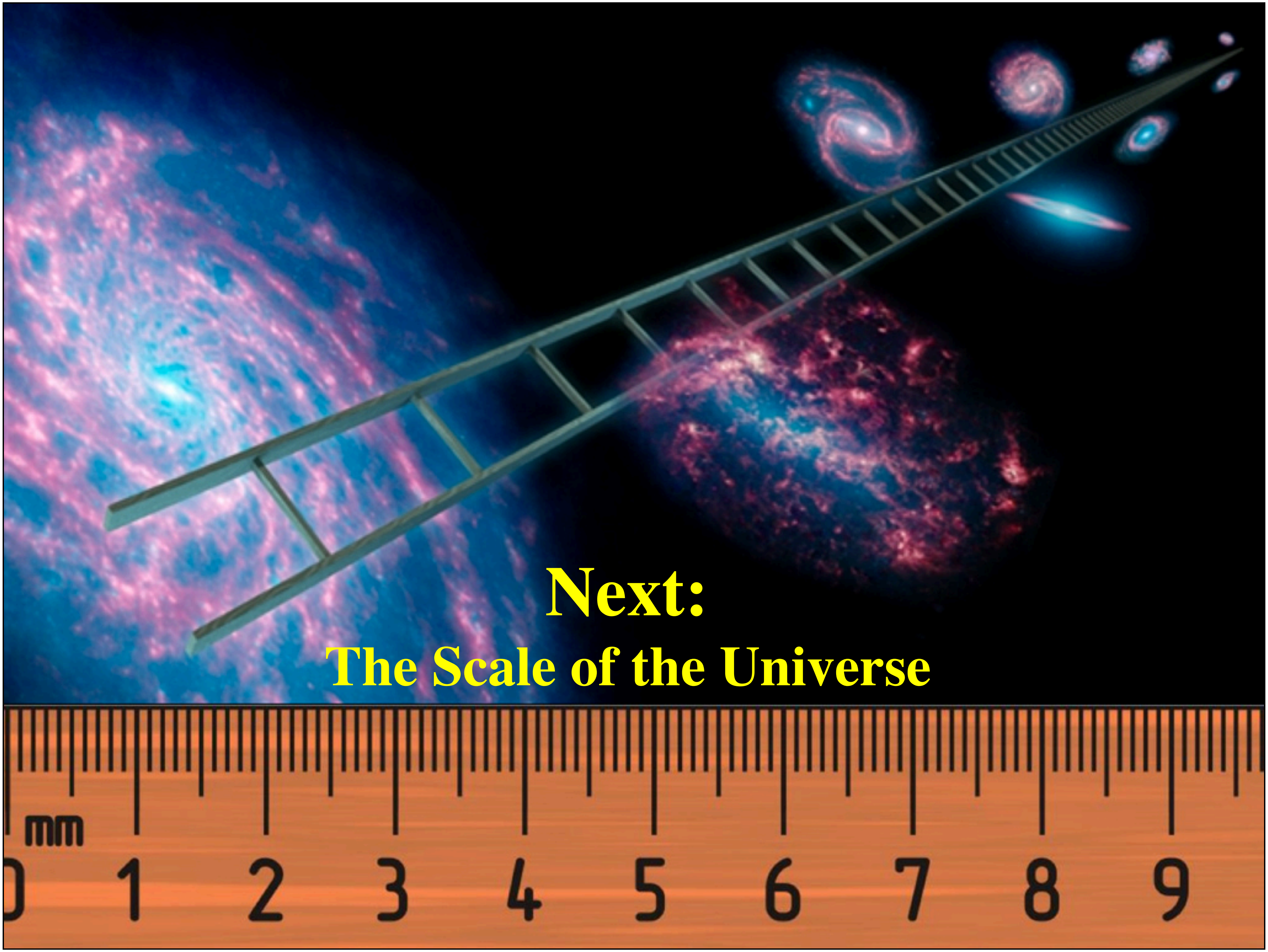


Cosmological Tests: Expected Generic Behavior of Various Models



Models with a lower density and/or positive Λ expand faster, are thus larger, older today, have more volume and thus higher source counts, at a given *z* sources are further away and thus appear fainter and smaller

Models with a higher density and lower Λ behave exactly the opposite


Next:
The Scale of the Universe
mm
1 2 3 4 5 6 7 8 9