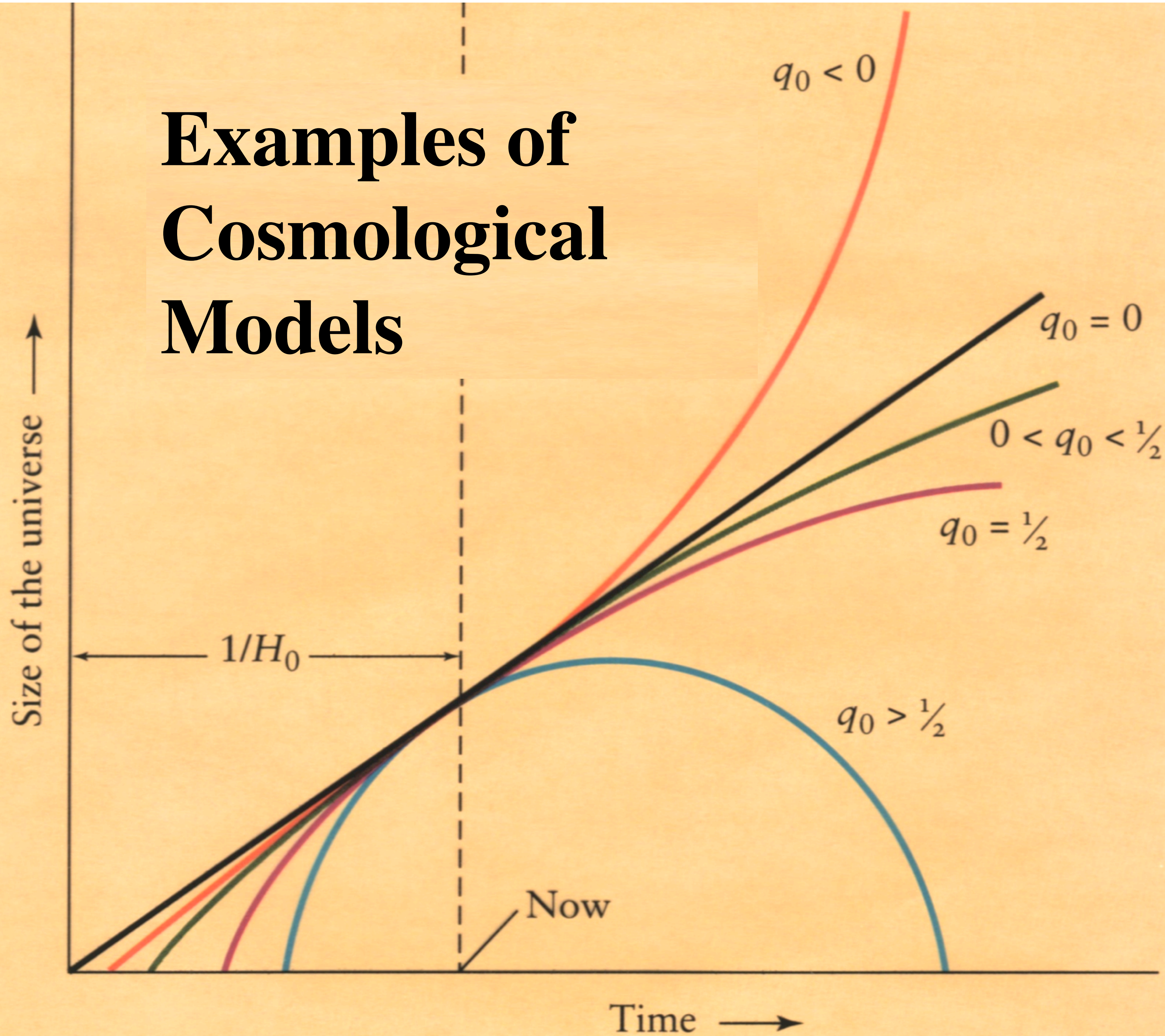
Examples of Cosmological Models



Specific Models

Consider several simple models:

- $\boldsymbol{k} = 0$, matter dominated, Einstein de Sitter
- $\boldsymbol{k} = 0$, radiation dominated
- $\boldsymbol{k} < 0, \rho = 0$, Milne Model
- $\boldsymbol{k} < 0, \rho > 0$
- $\boldsymbol{k} > 0$
- Λ dominated

$\boldsymbol{k} = 0$, matter dominated
Einstein de Sitter

Friedman Equation

with $\boldsymbol{k} = 0$ $\quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G\rho$

$$\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G\rho_0 a^{-3}$$

$$\Rightarrow \dot{a}^2 = \frac{8}{3}\pi G\rho_0 a^{-1}$$

$$\Rightarrow \frac{\partial a}{\partial t} = \pm\sqrt{\frac{8}{3}\pi G\rho_0}\, a^{-1/2}$$

$$\Rightarrow \int a^{1/2} da = \pm\sqrt{\frac{8}{3}\pi G\rho_0} \int dt$$

$$\Rightarrow a^{3/2} = \pm 2\sqrt{\frac{8}{3}\pi G\rho_0}\; t$$

$$\Rightarrow a = \pm 2\left(\frac{8}{3}\pi G\rho_0\right)^{1/3} t^{2/3}$$

i.e. $\quad \boxed{a \propto t^{2/3}}$

$k = 0$, radiation dominated

Friedman Equation:

with $k = 0$ $$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G\rho$$

$$\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G\rho_0 a^{-4}$$

$$\Rightarrow \dot{a} \propto a^{-1}$$

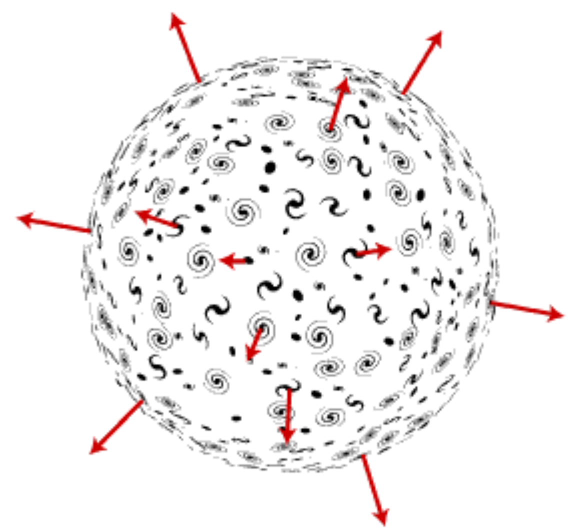
$$\Rightarrow \int a\,da \propto t$$

$$\Rightarrow a^2 \propto t$$

$$\boxed{\Rightarrow a \propto t^{1/2}}$$

Milne Cosmology

- Galaxies are created at one point in a flat spacetime
- Galaxies behave as massless test particles
  - then "trivially" get $v = H_0 d$ (faster particles move further)
- Special relativity holds
  - Lorentz contraction lets us have infinitely many galaxies (almost all right at the edge of our expanding bubble)
- No special locations
  - Milne was the first to state this "Cosmological Principle"



$\boldsymbol{k} = < 0, \rho = 0$
Milne Model

Friedman Equation:

with $\boldsymbol{\rho} = 0$ $$\left(\frac{\dot{a}}{a}\right)^2 = -\frac{kc^2}{a^2}$$

since $\boldsymbol{k} < 0$ $$\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 \propto \frac{+1}{a^2}$$

$$\Rightarrow \dot{a} = \text{const}$$

$$\boxed{\Rightarrow a \propto \pm t}$$

$\boldsymbol{k} < 0, \rho > 0$

Friedman Equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G\rho - \frac{kc^2}{a^2}$$

$$\dot{a}^2 > 0$$

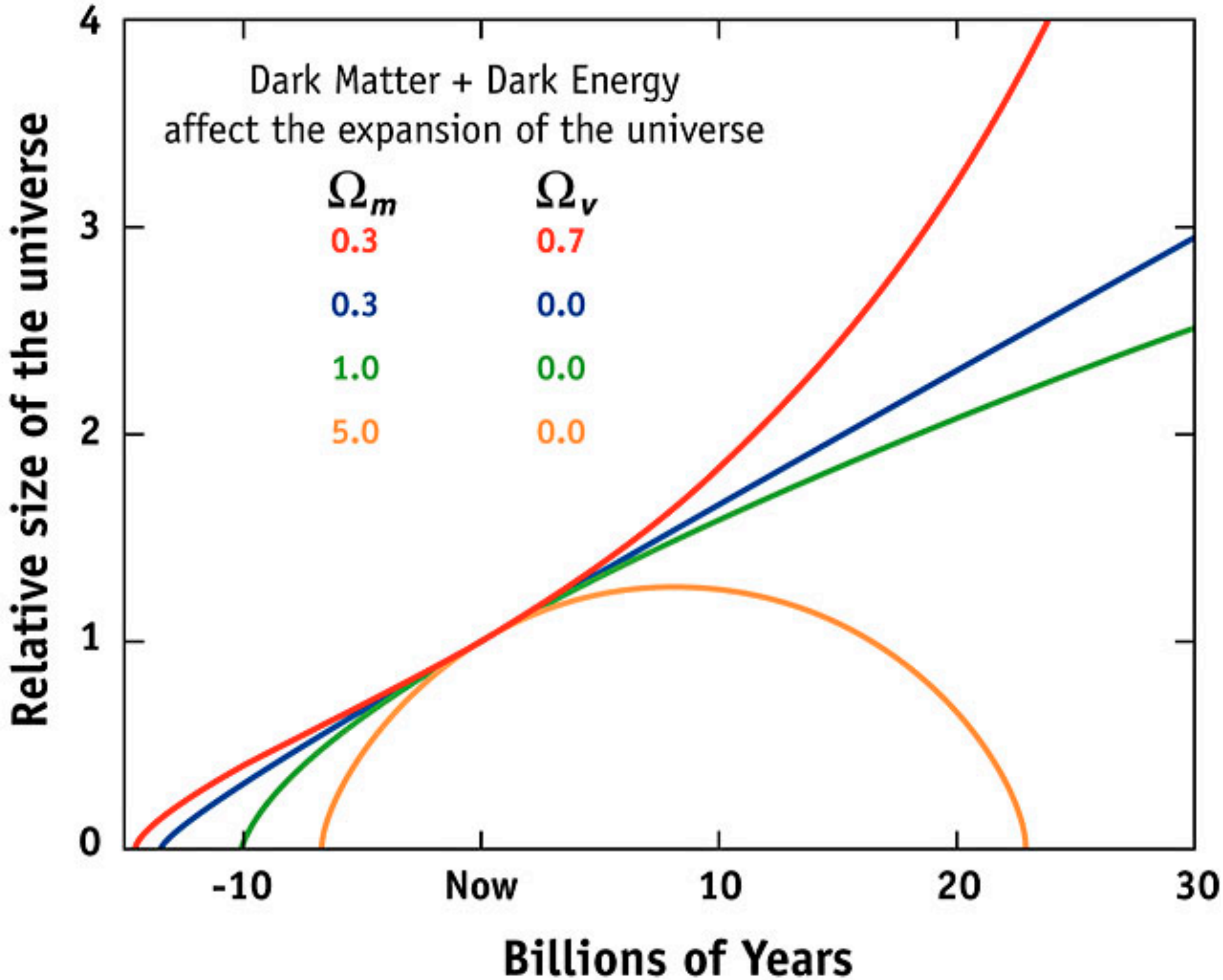
$$t \to \infty \quad a \to \infty, a^{-1} \to 0$$

$$\frac{1}{a^2} >> \frac{1}{a^3} >> \frac{1}{a^4}$$

either matter or radiation dominated models tend to Milne model

$$\boxed{\Rightarrow a \propto \pm t}$$

Matter Dominated Model



Positive Curvature Model: $k = +1$

Friedman Equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G\rho - \frac{kc^2}{a^2} \quad \Rightarrow \quad \dot{a}^2 = \frac{8}{3}\pi G\rho a^2 - c^2$$

if $\rho \propto a^{-3}$ or $\rho \propto a^{-4}$ then $\rho a^2 \propto a^{-n}$ $(n = 1,2)$ decreases

at some point $\dot{a}^2 = 0$

The acceleration equation is: $$\boxed{\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G\left(\rho + 3\frac{P}{c^2}\right)}$$

And since all other quantities are positive, $\ddot{a} < 0$

Therefore, a collapse is inevitable

Cosmological Constant (Λ) Dominated

Friedman Equation: $$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G\rho_\Lambda - \frac{kc^2}{a^2}$$

Which can be written as: $$\dot{a}^2 = C_0\rho a^2 - kc^2$$

Assuming that a is allowed to grow, then eventually

$C_0 a^2$ dominates over $-kc^2$ no matter what value of ***k***

If $\rho_\Lambda > 0$ then $\dot{a} > 0$

⇨ Universe expands for ever

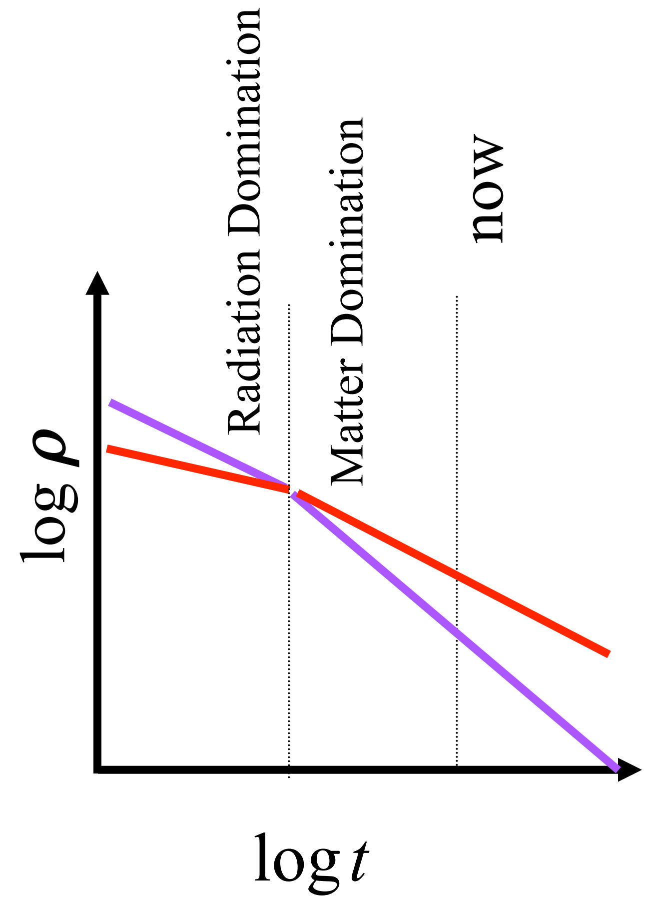
(Note that if $\rho_\Lambda < 0$ things get more complicated)

Models With Both Matter & Radiation

Harder to solve for ***ρ (t)***

However, to a good approximation, we can assume that $\boldsymbol{k} = 0$ and either radiation or matter dominate

	γ-dom	m-dom
$a(t)$	$\propto t^{1/2}$	$\propto t^{2/3}$
$\rho_{\mathrm{m}} \propto a^{-3}$	$\propto t^{-3/2}$	$\propto t^{-2}$
$\rho_{\gamma} \propto a^{-4}$	$\propto t^{-2}$	$\propto t^{-8/3}$



Generally, $$\frac{8\pi G\rho}{3} = H_0^{\;2}\left(\Omega_{\Lambda,0} + \Omega_{\mathrm{m},0}a^{-3} + \Omega_{\gamma,0}a^{-4}\right)$$

What is Dominant When?

Matter dominated ($w = 0$): $\rho \sim R^{-3}$

Radiation dominated ($w = 1/3$): $\rho \sim R^{-4}$

Dark energy ($w \sim -1$): $\rho \sim constant$

- Radiation density decreases the fastest with time
  - Must increase fastest on going back in time
  - Radiation must dominate early in the Universe
- Dark energy with $w \sim -1$ dominates last; it is the dominant component now, and in the (infinite?) future

Radiation domination → Matter domination → Dark energy domination

Dynamics of the Universe

$$R(t) \sim t^{2/[3(w+1)]}$$

- Matter dominated ($w = 0$): $R \sim t^{2/3}$
  - Decelerating
- Radiation dominated ($w = 1/3$): $R \sim t^{1/2}$
  - Decelerating
- Cosmological constant ($w = -1$): $R \sim e^{\lambda t}$ (special)
  - Accelerating
- Where is the transition?
  - $w > -1/3$ decelerating
  - $w < -1/3$ accelerating

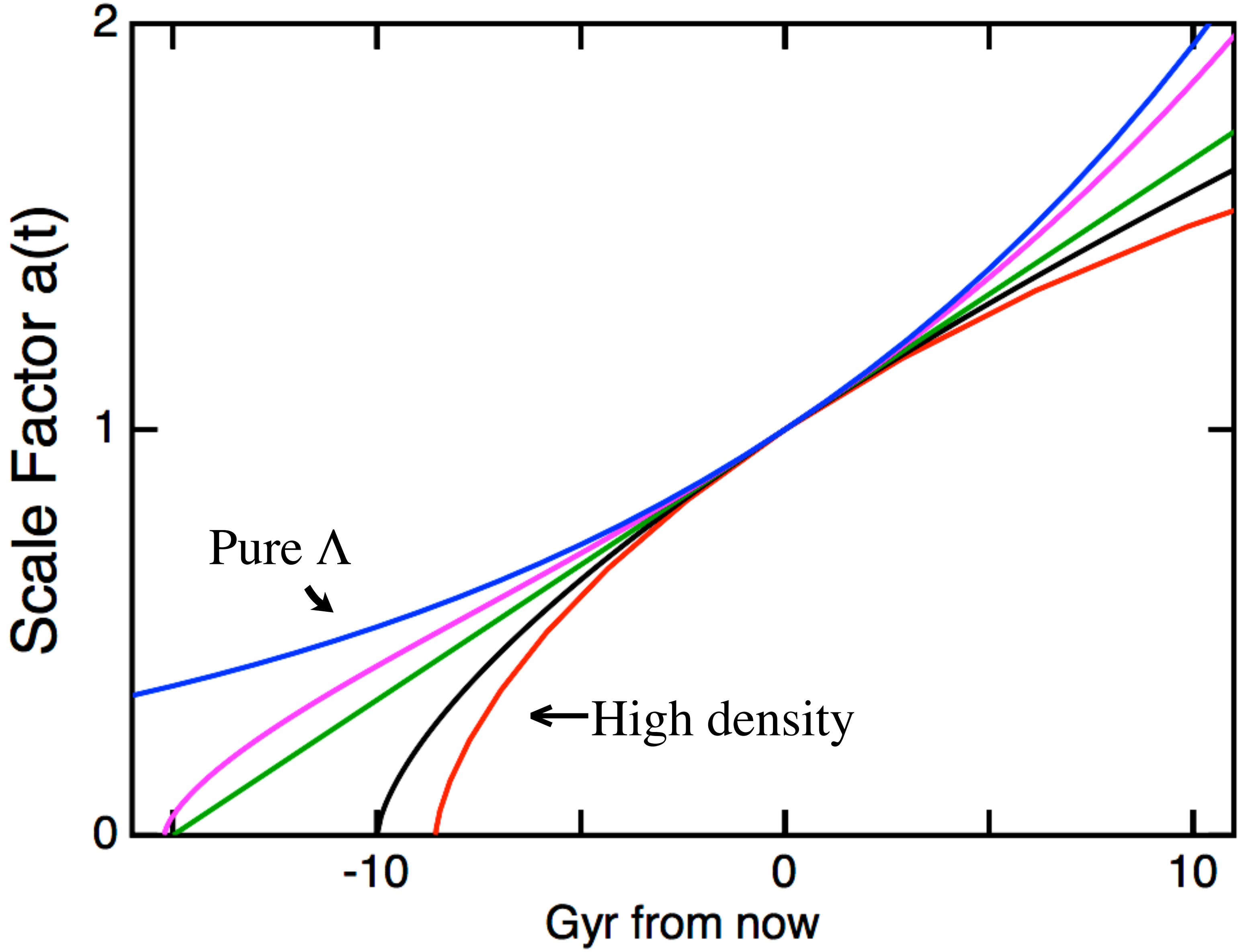


Fig. 10.— Scale factor *vs.* time for 5 different models: from top to bottom having $(\Omega_{m\circ}, \Omega_{v\circ}) = (0, 1)$ in blue, (0.25,0.75) in magenta, (0,0) in green, (1,0) in black and (2,0) in red. All have $H_\circ = 65$.

Next:
Distances in Cosmology