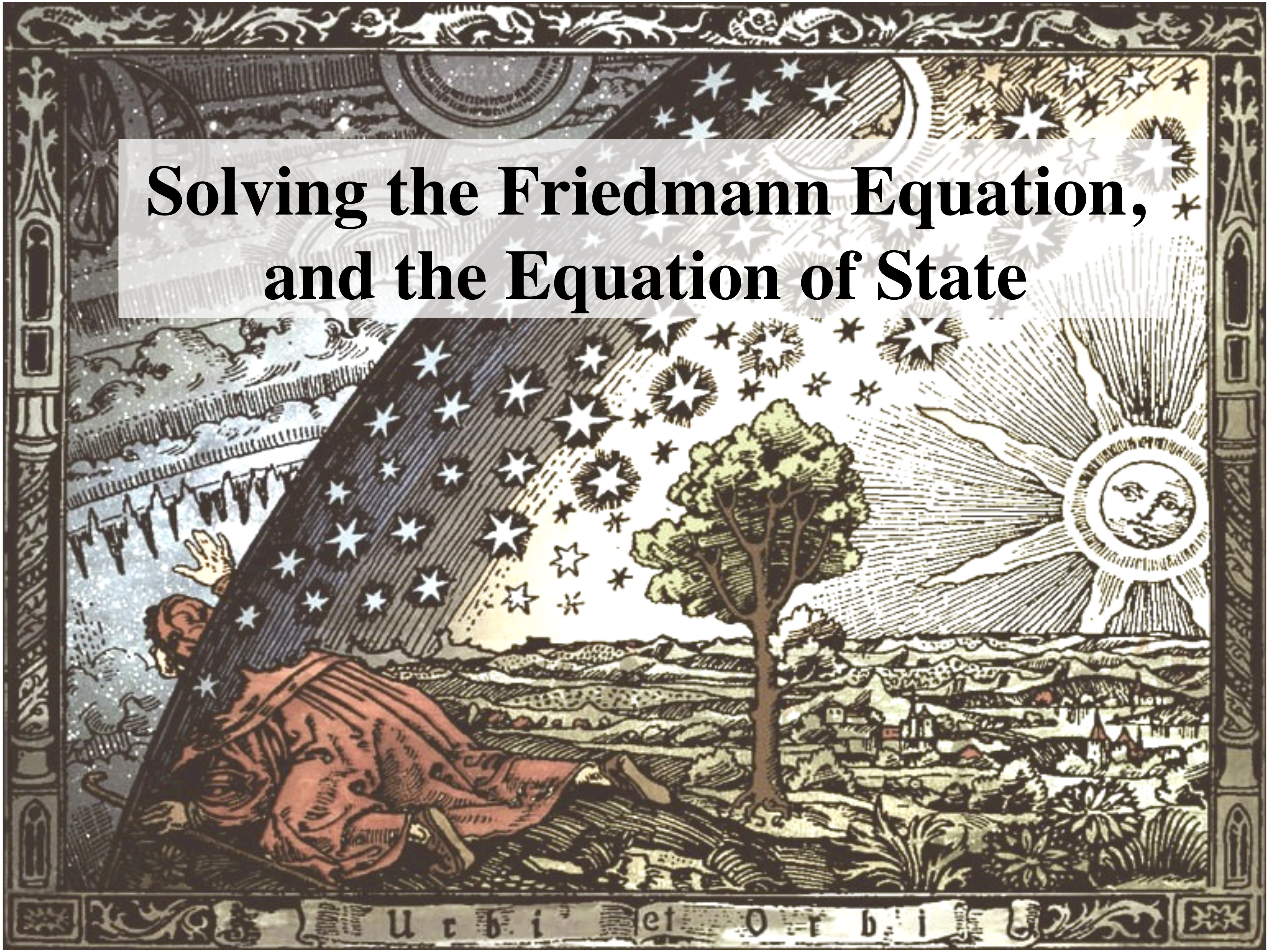
Solving the Friedmann Equation, and the Equation of State



Solving The Friedmann Equation

$$\boxed{\left(\frac{\dot{R}}{R}\right)^2 - \frac{8}{3}\pi G\rho - \frac{1}{3}\Lambda = -\frac{kc^2}{R^2}}$$

In order to solve it, we also need to define the behavior of mass/energy density $\rho(R)$ of any given mass/energy component, which is generally expressed through the **equation of state**, often written as a relation between pressure and density:

$$P = w\,\rho$$

w could be a constant, or a function of something

The EOS Parameter *W*

- Defined by the equation $\boldsymbol{p = w}\ \rho$
- Often called by itself the "equation of state"
  - Note: this is not necessarily the best way to describe the matter/energy density; it implies a fluid of some kind… This may be OK for the matter and radiation we know, but maybe it is not an optimal description for the dark energy
- Special values:

$\boldsymbol{w} = 0$ means $P = 0$, e.g., non-relativistic matter

$\boldsymbol{w} = 1/3$ is radiation or relativistic matter

$\boldsymbol{w} = -1$ looks just like a cosmological constant

… but it can have in principle any value, and it can be changing in redshift

Equation of State (EOS)

Some simple EOS we can consider:

- "Matter", "dust", "galaxies": $P = 0$
- Radiation: $P = \rho\, c^2 / 3$
- "Cosmological Constant": $P = -\rho\, c^2$

In reality, the universe contains a mix of these components, and maybe others as well…

Each will lead to a different evolution in redshift, and recall the basic GR paradigm:

Density determines the expansion

Expansion changes the density

Using the EOS

How does the matter/energy density change as the universe expands?

Start with the 3 cases we just described:

$$P = 0 \quad ①$$

$$P = \frac{\rho c^2}{3} \quad ②$$

$$P = -\rho c^2 \quad ③$$

We need to put these into the Fluid equation*:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0$$

* If you don't know what this is, please review your knowledge of the basic fluid mechanics, and note that in this context the cosmological expansion plays the role of the fluid flow

Using the EOS

$$P = 0 \quad 1$$
$$P = \frac{\rho c^2}{3} \quad 2$$
$$P = -\rho c^2 \quad 3$$

So for the 3 cases:

Substitute for P in the fluid equation:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0$$

1 $$\dot{\rho} + 3\frac{\dot{a}}{a}\rho = 0 \quad \Rightarrow \frac{1}{a^3}\frac{\partial}{\partial t}\left(\rho a^3\right) = 0 \quad \Rightarrow \rho \propto a^{-3}$$

2 $$\dot{\rho} + 4\frac{\dot{a}}{a}\rho = 0 \quad \Rightarrow \frac{1}{a^4}\frac{\partial}{\partial t}\left(\rho a^4\right) = 0 \quad \Rightarrow \rho \propto a^{-4}$$

3 $$\dot{\rho} = 0 \quad \Rightarrow \rho \text{ const}$$

Evolution of the Density

Generally, $\rho \sim R^{-3(w+1)}$

- Matter dominated ($w = 0$): $\rho \sim R^{-3}$
- Radiation dominated ($w = 1/3$): $\rho \sim R^{-4}$
- Cosmological constant ($w = -1$): $\rho = constant$
- Dark energy with $w < -1$ e.g., $w = -2$: $\rho \sim R^{+3}$
  - Energy density *increases* as is stretched out!
  - Eventually would dominate over even the energies holding atoms together! ("Big Rip")

In a mixed universe, different components will dominate the global dynamics at different times

Note also that in principle, w could be a function of time, density, etc.

Next:

Examples of Cosmological Models