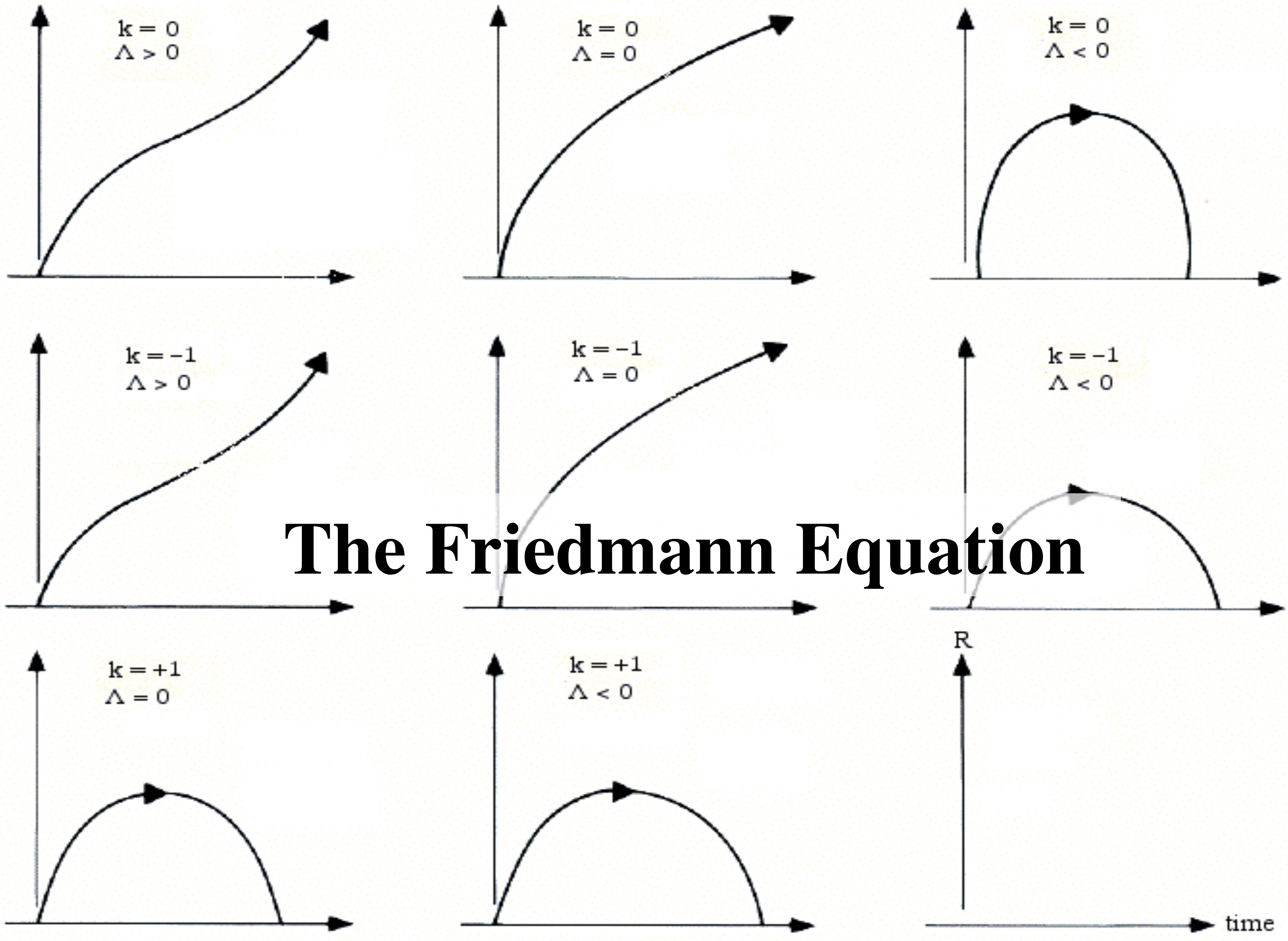
The Friedmann Equation



Deriving the Friedmann Equation

Can derive the evolution of ***R(t)*** using mostly Newtonian mechanics, provided we accept two results from General Relativity:

1) **Birkhoff's theorem**: for a spherically symmetric system, the force due to gravity at radius r is determined only by the mass *interior* to that radius.
2) **Energy** contributes to the gravitating mass density, which equals:

$$\rho_m + \frac{u}{c^2}$$

energy density (ergs cm^{-3}) of radiation and relativistic particles

density of matter

Deriving the Friedmann Equation

Start with a test particle on the surface of an expanding sphere of radius R. Its equation of motion is

$$\ddot{R} = -\frac{4\pi}{3} G\rho R$$

Since density is proportional to R^{-3}, and we define "now" with a 0 subscript, and $R_0=1$, we have

$$\rho = \rho_0 R^{-3}$$

Which we can insert into the equation of motion to get

$$\ddot{R} = -\frac{4\pi}{3} \frac{G\rho_0}{R^2}$$

Note that if rho$_0$ is nonzero, the Universe must be expanding or contracting. It cannot be static.

How do we integrate this? Multiply both side by Rdot to get

$$\dot{R}\ddot{R} + \frac{4\pi}{3}\frac{G\rho_0}{R^2}\dot{R} = 0$$

And remember that

$$d(\dot{R}^2)/dt = 2\dot{R}\ddot{R}$$

So that

$$\frac{1}{2}\frac{d(\dot{R}^2)}{dt} + \frac{4\pi G\rho_0}{3}\frac{1}{R^2}\frac{dR}{dt} = 0$$

Now, also remember:

$$\frac{1}{R^2}\frac{dR}{dt} = -\frac{d(1/R)}{dt}$$

So that we have

$$\frac{d}{dt}\left[\dot{R}^2 - \frac{(8\pi G\rho_0/3)}{R}\right] = 0$$

Which means that the expression inside the brackets must be constant

$$\dot{R}^2 - \frac{(8\pi G\rho_0/3)}{R} = k$$

Replacing rho_0 with rho, and dividing by R^2,

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8}{3}\pi G\rho = -\frac{k}{R^2}$$

And finally: $$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8}{3}\pi G\rho = -\frac{k}{R^2}$$

What does this mean?

- If **k=0**, then Rdot is always positive, and the expansion continues at an ever slowing pace (since rho is dropping). This is called a **critical or flat universe**.

- If **k>0**, Rdot is initially positive, but will reach a point where it changes sign. Expansion turns into contraction. This is a **closed universe**.

- If **k<0**, Rdot is always positive, and never goes to zero -- expansion always continues. This is an **open universe**.

The Friedmann Equation

A more complete derivation, including the cosmological constant term, gives:

$$\boxed{\left(\frac{\dot{R}}{R}\right)^2 - \frac{8}{3}\pi G\rho - \frac{1}{3}\Lambda = -\frac{kc^2}{R^2}}$$

The Friedmann Eqn. is effectively *the equation of motion for a relativistic, homogeneous, isotropic universe*.

In order to derive cosmological models from it, we also need to specify *the equation of state* of the "cosmological fluid" which fills the universe.

Next:
Cosmological Parameters