

# Quartersquares: an old-fashioned way to multiply

Recall the fact that

$$ab = \frac{(a+b)^2}{4} - \frac{(a-b)^2}{4}.$$

To multiply  $a$  and  $b$ ,

- Look up  $a + b$  in the table.
- Look up  $a - b$  in the table.
- Subtract the quartersquare of  $a - b$  from the quartersquare of  $a + b$ .

From a complexity point of view, this is an improvement: adding (subtracting) an  $n$  digit number to (from) an  $n$  digit number requires  $n$  operations, but multiplying an  $n$  digit number by an  $n$  digit number using the usual algorithm requires  $n^2$  operations. **Multiplication is expensive.** By using quartersquares, we can replace expensive multiplications with cheaper additions, subtractions, and table lookups.

| $n$ | $n^2/4$          | $n$ | $n^2/4$          | $n$ | $n^2/4$           | $n$ | $n^2/4$           | $n$ | $n^2/4$           | $n$ | $n^2/4$           |
|-----|------------------|-----|------------------|-----|-------------------|-----|-------------------|-----|-------------------|-----|-------------------|
| 1   | $\frac{1}{4}$    | 26  | 169              | 51  | $650\frac{1}{4}$  | 76  | 1444              | 101 | $2550\frac{1}{4}$ | 126 | 3969              |
| 2   | 1                | 27  | $182\frac{1}{4}$ | 52  | 676               | 77  | $1482\frac{1}{4}$ | 102 | 2601              | 127 | $4032\frac{1}{4}$ |
| 3   | $2\frac{1}{4}$   | 28  | 196              | 53  | $702\frac{1}{4}$  | 78  | 1521              | 103 | $2652\frac{1}{4}$ | 128 | 4096              |
| 4   | 4                | 29  | $210\frac{1}{4}$ | 54  | 729               | 79  | $1560\frac{1}{4}$ | 104 | 2704              | 129 | $4160\frac{1}{4}$ |
| 5   | $6\frac{1}{4}$   | 30  | 225              | 55  | $756\frac{1}{4}$  | 80  | 1600              | 105 | $2756\frac{1}{4}$ | 130 | 4225              |
| 6   | 9                | 31  | $240\frac{1}{4}$ | 56  | 784               | 81  | $1640\frac{1}{4}$ | 106 | 2809              | 131 | $4290\frac{1}{4}$ |
| 7   | $12\frac{1}{4}$  | 32  | 256              | 57  | $812\frac{1}{4}$  | 82  | 1681              | 107 | $2862\frac{1}{4}$ | 132 | 4356              |
| 8   | 16               | 33  | $272\frac{1}{4}$ | 58  | 841               | 83  | $1722\frac{1}{4}$ | 108 | 2916              | 133 | $4422\frac{1}{4}$ |
| 9   | $20\frac{1}{4}$  | 34  | 289              | 59  | $870\frac{1}{4}$  | 84  | 1764              | 109 | $2970\frac{1}{4}$ | 134 | 4489              |
| 10  | 25               | 35  | $306\frac{1}{4}$ | 60  | 900               | 85  | $1806\frac{1}{4}$ | 110 | 3025              | 135 | $4556\frac{1}{4}$ |
| 11  | $30\frac{1}{4}$  | 36  | 324              | 61  | $930\frac{1}{4}$  | 86  | 1849              | 111 | $3080\frac{1}{4}$ | 136 | 4624              |
| 12  | 36               | 37  | $342\frac{1}{4}$ | 62  | 961               | 87  | $1892\frac{1}{4}$ | 112 | 3136              | 137 | $4692\frac{1}{4}$ |
| 13  | $42\frac{1}{4}$  | 38  | 361              | 63  | $992\frac{1}{4}$  | 88  | 1936              | 113 | $3192\frac{1}{4}$ | 138 | 4761              |
| 14  | 49               | 39  | $380\frac{1}{4}$ | 64  | 1024              | 89  | $1980\frac{1}{4}$ | 114 | 3249              | 139 | $4830\frac{1}{4}$ |
| 15  | $56\frac{1}{4}$  | 40  | 400              | 65  | $1056\frac{1}{4}$ | 90  | 2025              | 115 | $3306\frac{1}{4}$ | 140 | 4900              |
| 16  | 64               | 41  | $420\frac{1}{4}$ | 66  | 1089              | 91  | $2070\frac{1}{4}$ | 116 | 3364              | 141 | $4970\frac{1}{4}$ |
| 17  | $72\frac{1}{4}$  | 42  | 441              | 67  | $1122\frac{1}{4}$ | 92  | 2116              | 117 | $3422\frac{1}{4}$ | 142 | 5041              |
| 18  | 81               | 43  | $462\frac{1}{4}$ | 68  | 1156              | 93  | $2162\frac{1}{4}$ | 118 | 3481              | 143 | $5112\frac{1}{4}$ |
| 19  | $90\frac{1}{4}$  | 44  | 484              | 69  | $1190\frac{1}{4}$ | 94  | 2209              | 119 | $3540\frac{1}{4}$ | 144 | 5184              |
| 20  | 100              | 45  | $506\frac{1}{4}$ | 70  | 1225              | 95  | $2256\frac{1}{4}$ | 120 | 3600              | 145 | $5256\frac{1}{4}$ |
| 21  | $110\frac{1}{4}$ | 46  | 529              | 71  | $1260\frac{1}{4}$ | 96  | 2304              | 121 | $3660\frac{1}{4}$ | 146 | 5329              |
| 22  | 121              | 47  | $552\frac{1}{4}$ | 72  | 1296              | 97  | $2352\frac{1}{4}$ | 122 | 3721              | 147 | $5402\frac{1}{4}$ |
| 23  | $132\frac{1}{4}$ | 48  | 576              | 73  | $1332\frac{1}{4}$ | 98  | 2401              | 123 | $3782\frac{1}{4}$ | 148 | 5476              |
| 24  | 144              | 49  | $600\frac{1}{4}$ | 74  | 1369              | 99  | $2450\frac{1}{4}$ | 124 | 3844              | 149 | $5550\frac{1}{4}$ |
| 25  | $156\frac{1}{4}$ | 50  | 625              | 75  | $1406\frac{1}{4}$ | 100 | 2500              | 125 | $3906\frac{1}{4}$ | 150 | 5625              |