Undecidability

Everything is an Integer

Countable and Uncountable Sets

Turing Machines

Recursive and Recursively Enumerable Languages

Integers, Strings, and Other Things

- Data types have become very important as a programming tool.
- But at another level, there is only one type, which you may think of as integers or strings.
- Key point: Strings that are programs are just another way to think about the same one data type.

Example: Text

- Strings of ASCII or Unicode characters can be thought of as binary strings, with 8 or 16 bits/character.
- Binary strings can be thought of as integers.
- It makes sense to talk about “the i-th string.”

Binary Strings to Integers

- There's a small glitch:
  - If you think simply of binary integers, then strings like 101, 0101, 00101,... all appear to be "the fifth string."
- Fix by prepending a "1" to the string before converting to an integer.
  - Thus, 101, 0101, and 00101 are the 13^{th}, 21^{st}, and 37^{th} strings, respectively.

Example: Images

- Represent an image in (say) GIF.
- The GIF file is an ASCII string.
- Convert string to binary.
- Convert binary string to integer.
- Now we have a notion of “the i-th image.”

Example: Proofs

- A formal proof is a sequence of logical expressions, each of which follows from the ones before it.
- Encode mathematical expressions of any kind in Unicode.
- Convert expression to a binary string and then an integer.

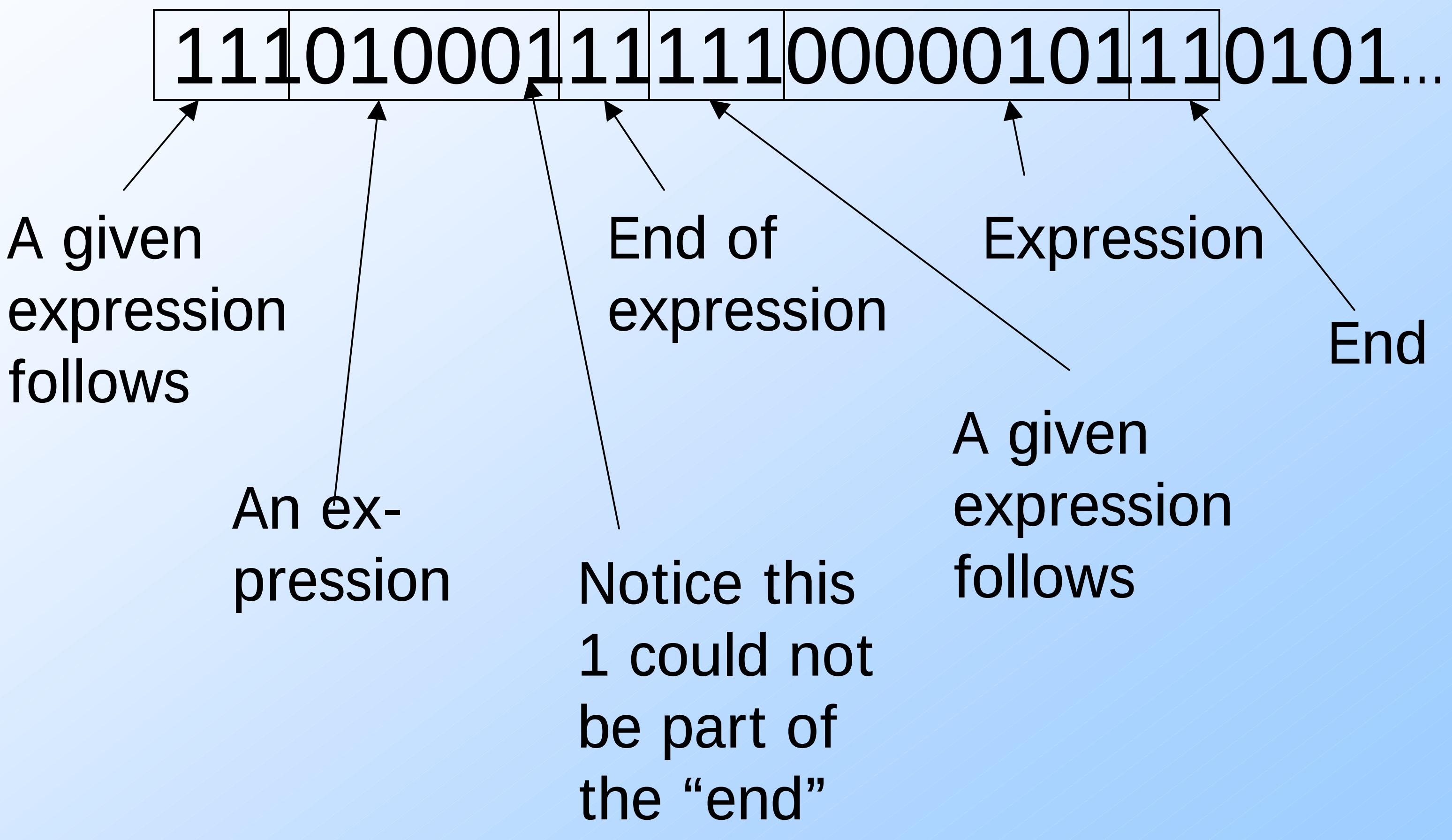
Proofs – (2)

- But a proof is a sequence of expressions, so we need a way to separate them.
- Also, we need to indicate which expressions are given and which follow from previous expressions.

Proofs – (3)

- Quick-and-dirty way to introduce new symbols into binary strings:
  1. Given a binary string, precede each bit by 0.
     - Example: 101 becomes 010001.
  2. Use strings of two or more 1's as the special symbols.
     - Example: 111 = "the following expression is given"; 11 = "end of expression."

Example: Encoding Proofs



Example: Programs

- Programs are just another kind of data.
- Represent a program in ASCII.
- Convert to a binary string, then to an integer.
- Thus, it makes sense to talk about "the i-th program."
- Hmm...There aren't all that many programs.

Finite Sets

- A *finite set* has a particular integer that is the count of the number of members.
- Example: {a, b, c} is a finite set; its *cardinality* is 3.
- It is impossible to find a 1-1 mapping between a finite set and a proper subset of itself.

Infinite Sets

- Formally, an *infinite set* is a set for which there is a 1-1 correspondence between itself and a proper subset of itself.
- Example: the positive integers {1, 2, 3,...} is an infinite set.
  - There is a 1-1 correspondence 1<->2, 2<->4, 3<->6,... between this set and a proper subset (the set of even integers).

Countable Sets

- A *countable set* is a set with a 1-1 correspondence with the positive integers.
  - Hence, all countable sets are infinite.
- Example: All integers.
  - 0<->1; -i <-> 2i; +i <-> 2i+1.
  - Thus, order is 0, -1, 1, -2, 2, -3, 3,...
- Examples: set of binary strings, set of Java programs.

Example: Pairs of Integers

- Order the pairs of positive integers first by sum, then by first component:
- [1,1], [2,1], [1,2], [3,1], [2,2], [1,3], [4,1], [3,2],..., [1,4], [5,1],...
- Interesting exercise: figure out the function f(i,j) such that the pair [i,j] corresponds to the integer f(i,j) in this order.

Enumerations

- An *enumeration* of a set is a 1-1 correspondence between the set and the positive integers.
- Thus, we have seen enumerations for strings, programs, proofs, and pairs of integers.

How Many Languages?

- Are the languages over {0,1} countable?
- No; here's a proof.
- Suppose we could enumerate all languages over {0,1} and talk about "the i-th language."
- Consider the language L = { w | w is the i-th binary string and w is not in the i-th language}.

Proof – Continued

- Clearly, L is a language over {0,1}.
- Thus, it is the j-th language for some particular j.
- Let x be the j-th string.
- Is x in L?
  - If so, x is not in L by definition of L.
  - If not, then x is in L by definition of L.

L_j

x

Recall: L = { w | w is the i-th binary string and w is not in the i-th language}.

j-th

Proof – Concluded

- We have a contradiction: x is neither in L nor not in L, so our sole assumption (that there was an enumeration of the languages) is wrong.
- Comment: This is really bad; there are more languages than programs.
- E.g., there are languages with no membership algorithm.

Diagonalization Picture

Strings

Languages

	1	2	3	4	5	...
1	1	0	1	1	0	...
2		1				
3			0			
4				0		
5					1	
...						...

Diagonalization Picture

Flip each diagonal entry

Strings

Languages	1	2	3	4	5	...
1	0	0	1	1	0	...
2		0				
3			1			
4				1		
5					0	
...						...

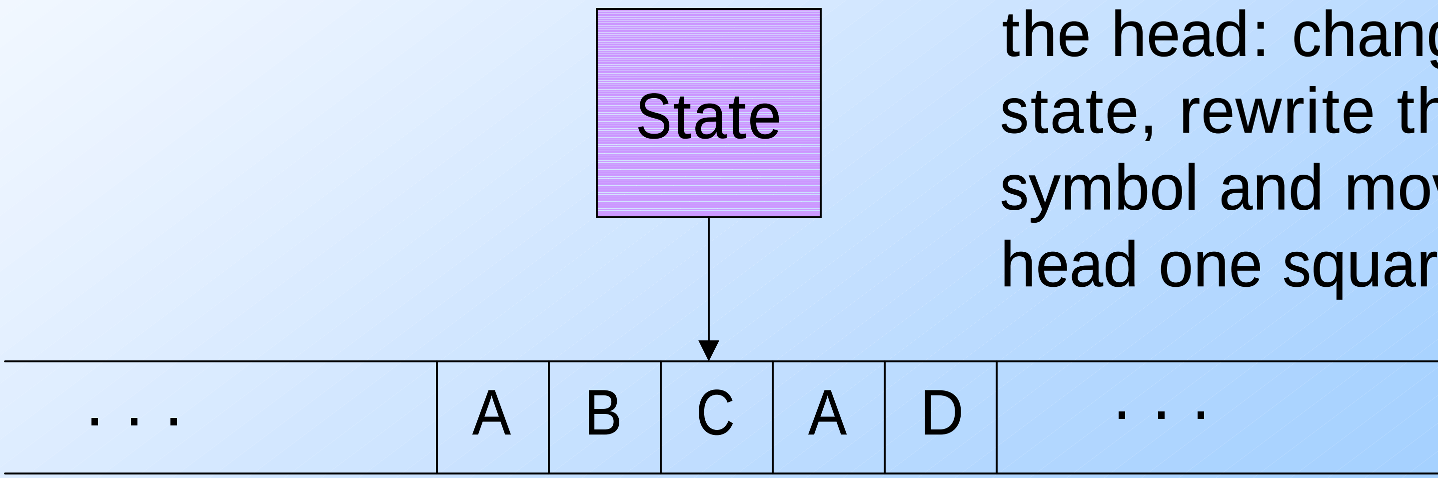
Can't be a row – it disagrees in an entry of each row.

Turing-Machine Theory

- The purpose of the theory of Turing machines is to prove that certain specific languages have no algorithm.
- Start with a language about Turing machines themselves.
- Reductions are used to prove more common questions undecidable.

Picture of a Turing Machine

Action: based on the state and the tape symbol under the head: change state, rewrite the symbol and move the head one square.



Infinite tape with squares containing tape symbols chosen from a finite alphabet

Why Turing Machines?

- Why not deal with C programs or something like that?
- Answer: You can, but it is easier to prove things about TM's, because they are so simple.
  - And yet they are as powerful as any computer.
    - More so, in fact, since they have infinite memory.

Turing-Machine Formalism

- A TM is described by:
  1. A finite set of *states* (Q, typically).
  2. An *input alphabet* (Σ, typically).
  3. A *tape alphabet* (Γ, typically; contains Σ).
  4. A *transition function* (δ, typically).
  5. A *start state* (q_0, in Q, typically).
  6. A *blank symbol* (B, in Γ− Σ, typically).
     - All tape except for the input is blank initially.
  7. A set of *final states* (F ⊆ Q, typically).

Conventions

- a, b, ... are input symbols.
- ..., X, Y, Z are tape symbols.
- ..., w, x, y, z are strings of input symbols.
- α, β,... are strings of tape symbols.

The Transition Function

- Takes two arguments:
  1. A state, in Q.
  2. A tape symbol in Γ.
- δ(q, Z) is either undefined or a triple of the form (p, Y, D).
  - p is a state.
  - Y is the new tape symbol.
  - D is a *direction*, L or R.

Example: Turing Machine

- This TM scans its input right, looking for a 1.
- If it finds one, it changes it to a 0, goes to final state f, and halts.
- If it reaches a blank, it changes it to a 1 and moves left.

Example: Turing Machine – (2)

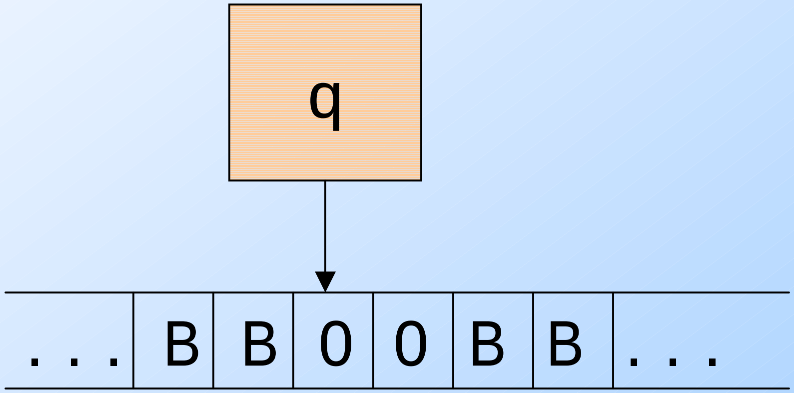
- States = {q (start), f (final)}.
- Input symbols = {0, 1}.
- Tape symbols = {0, 1, B}.
- $\delta(q, 0) = (q, 0, R)$.
- $\delta(q, 1) = (f, 0, R)$.
- $\delta(q, B) = (q, 1, L)$.

Simulation of TM

$\delta(q, 0) = (q, 0, R)$

$\delta(q, 1) = (f, 0, R)$

$\delta(q, B) = (q, 1, L)$



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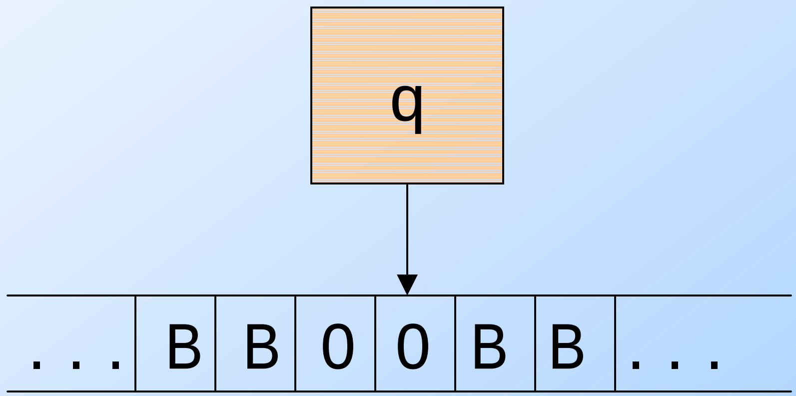

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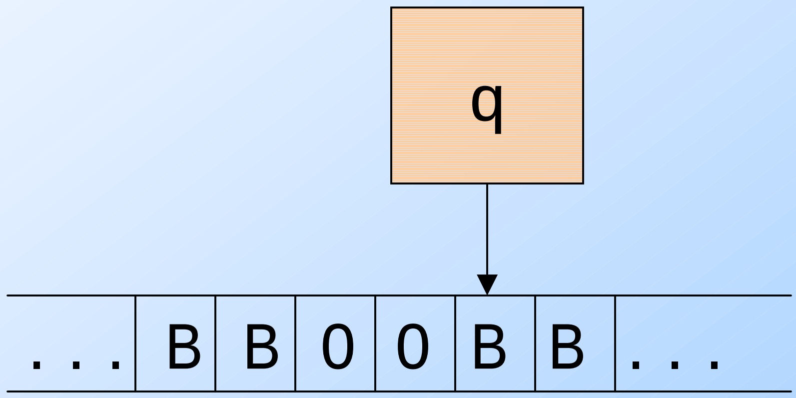

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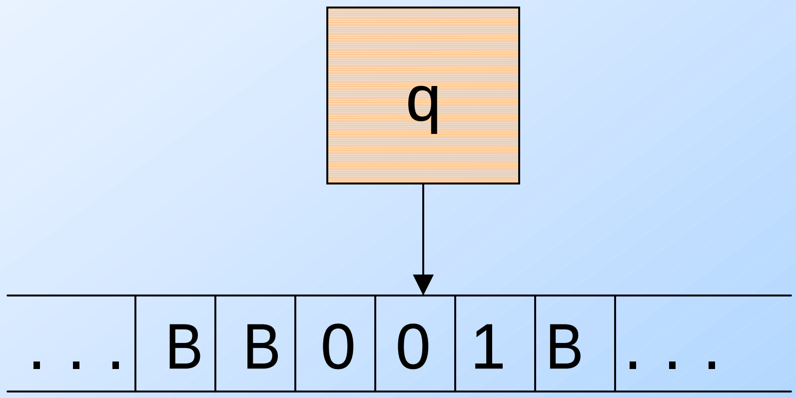

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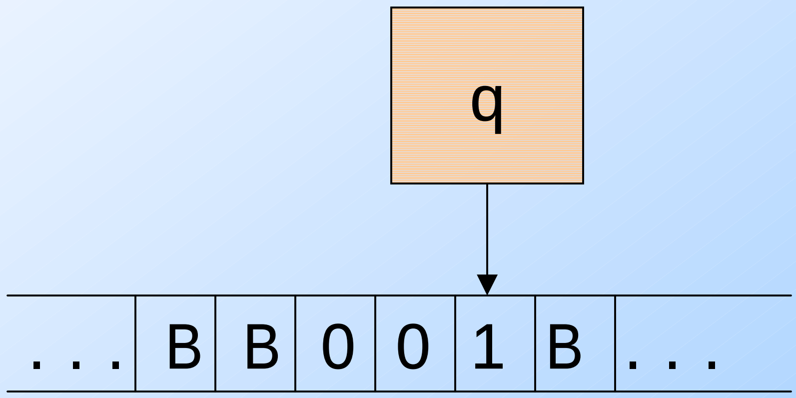

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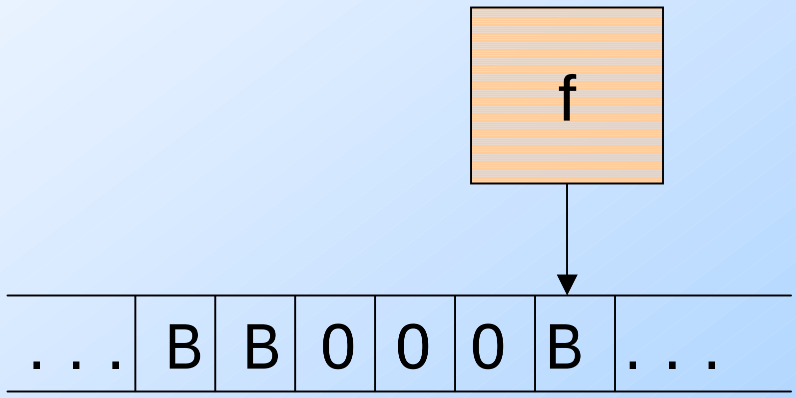

No move is possible. The TM halts and accepts.

Instantaneous Descriptions of a Turing Machine

- Initially, a TM has a tape consisting of a string of input symbols surrounded by an infinity of blanks in both directions.
- The TM is in the start state, and the head is at the leftmost input symbol.

TM ID's – (2)

- An ID is a string $\alpha q\beta$, where $\alpha\beta$ includes the tape between the leftmost and rightmost nonblanks.
- The state q is immediately to the left of the tape symbol scanned.
- If q is at the right end, it is scanning B.
  - If q is scanning a B at the left end, then consecutive B's at and to the right of q are part of α.

TM ID's – (3)

- As for PDA's we may use symbols ⊦ and ⊦* to represent "becomes in one move" and "becomes in zero or more moves," respectively, on ID's.
- Example: The moves of the previous TM are q00⊦0q0⊦00q⊦0q01⊦00q1⊦000f

Formal Definition of Moves

1. If $\delta(q, Z) = (p, Y, R)$, then
   - $\alpha qZ\beta \vdash \alpha Yp\beta$
   - If Z is the blank B, then also $\alpha q \vdash \alpha Yp$
2. If $\delta(q, Z) = (p, Y, L)$, then
   - For any X, $\alpha XqZ\beta \vdash \alpha pXY\beta$
   - In addition, $qZ\beta \vdash pBY\beta$

Languages of a TM

- A TM defines a language by final state, as usual.
- $L(M) = \{w \mid q_0w \vdash^* I$, where I is an ID with a final state$\}$.
- Or, a TM can accept a language by halting.
- $H(M) = \{w \mid q_0w \vdash^* I$, and there is no move possible from ID I$\}$.

Equivalence of Accepting and Halting

1. If L = L(M), then there is a TM M' such that L = H(M').
2. If L = H(M), then there is a TM M" such that L = L(M").

Proof of 1: Final State -> Halting

- Modify M to become M' as follows:
  1. For each final state of M, remove any moves, so M' halts in that state.
  2. Avoid having M' accidentally halt.
     - Introduce a new state s, which runs to the right forever; that is $\delta(s, X) = (s, X, R)$ for all symbols X.
     - If q is not a final state, and $\delta(q, X)$ is undefined, let $\delta(q, X) = (s, X, R)$.

Proof of 2: Halting -> Final State

- Modify M to become M" as follows:
  1. Introduce a new state f, the only final state of M".
  2. f has no moves.
  3. If $\delta(q, X)$ is undefined for any state q and symbol X, define it by $\delta(q, X) = (f, X, R)$.

Recursively Enumerable Languages

- We now see that the classes of languages defined by TM's using final state and halting are the same.
- This class of languages is called the *recursively enumerable languages*.
  - Why? The term actually predates the Turing machine and refers to another notion of computation of functions.

Recursive Languages

- An *algorithm* is a TM, accepting by final state, that is guaranteed to halt whether or not it accepts.
- If L = L(M) for some TM M that is an algorithm, we say L is a *recursive language*.
  - Why? Again, don't ask; it is a term with a history.

Example: Recursive Languages

- Every CFL is a recursive language.
  - Use the CYK algorithm.
- Almost anything you can think of is recursive.