Algorithms: Design and Analysis, Part II

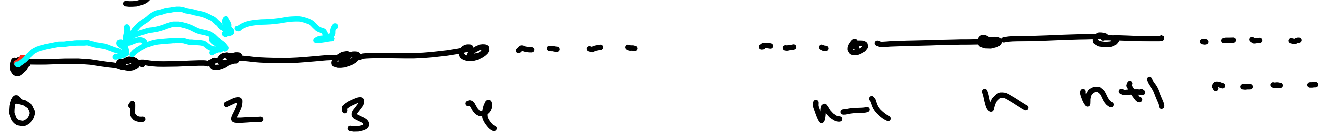
Local Search

Random Walks on a Line

Random Walks

Key to analyzing Papadimitriou's algorithm:
random walks on the non negative integers (trust me!)

Setup: Initially (at time 0), at position 0.



At each time step, your position goes up or down by 1, with 50/50 probability.

[except if at position 0, in which case you move to position 1 with 100% probability]

Quiz

Notation: for an integer $n \geq 0$, let

T_n = number of steps until random walk reaches position n.

[a random variable, sample space = coin flips at all time steps]

Question: What is $E[T_n]$? (your best guess)

(A) $\Theta(n)$

(B) $\Theta(n^2)$

(C) $\Theta(n^3)$

(D) $\Theta(2^n)$

Coming up: $E[T_n] = n^2$.

Analysis of T_n

Let Z_i = number of random walks steps to get to n from i.

(note $Z_0 = T_n$)

Edge cases: $E[Z_n] = 0$.

$$E[Z_0] = 1 + E[Z_1]$$

For $i \in \{1,2,3,\ldots,n-1\}$:

$$E[Z_i] = \overbrace{\Pr[\text{go left}]}^{\frac{1}{2}} \cdot \overbrace{E[Z_i \mid \text{go left}]}^{(1+E[Z_{i-1}])} + \overbrace{\Pr[\text{go right}]}^{\frac{1}{2}} \cdot \overbrace{E[Z_i \mid \text{go right}]}^{(1+E[Z_{i+1}])}$$

$$= 1 + \tfrac{1}{2}E[Z_{i+1}] + \tfrac{1}{2}E[Z_{i-1}]$$

Rearranging: $E[Z_i] - E[Z_{i+1}] = E[Z_{i-1}] - E[Z_i] + 2$

Finishing the Proof of Claim

So:

$$\begin{aligned}
E[Z_0] - \cancel{E[Z_1]} &= 1 \\
\cancel{E[Z_1]} - \cancel{E[Z_2]} &= 3 \\
\cancel{E[Z_2]} - \cancel{E[Z_3]} &= 5 \\
&\vdots \\
+\ \cancel{E[Z_{n-1}]} - \cancelto{0}{E[Z_n]} &= 2n-1 \\
\hline
E[Z_0] &= n^2
\end{aligned}$$

$$E[Z_0] = E[T_n]$$

$\frac{n}{2}$ pairs of numbers, each sums to $2n$

QED!

A Corollary

Corollary: $\Pr[T_n > 2n^2] \leq \frac{1}{2}$.

(special case of Markov's inequality)

Proof: Let p denote $\Pr[T_n > 2n^2]$.

We have $n^2 = E[T_n] = \underbrace{\sum_{k=0}^{2n^2} k \cdot \Pr[T_n = k]}_{\geq 0} + \sum_{k=2n^2+1}^{\infty} \underbrace{k}_{\geq 2n^2} \cdot \Pr[T_n = k]$

by last claim

$$\geq 2n^2 \cdot \Pr[T_n > 2n^2]$$

$$= 2n^2 p$$

$\Rightarrow p \leq \frac{1}{2}$

QED!