Algorithms: Design and Analysis, Part II

Approximation Algorithms for NP-Complete Problems

Analysis of a Greedy Knapsack Heuristic

Performance Guarantee

Theorem: value of the 3-step greedy algorithm's solution is always $\geq 50\%\cdot$ value of an optimal solution.

Thought experiment: what if we were allowed to fill fully the knapsack using a suitable "fraction" (like 70%) of item (k+1)? [the value of which is "pro-rated"]

⇒ will call this the "greedy fractional solution"

Example: $W=3$, $v_1=3$, $v_2=2$, $w_1=w_2=2$ (get 100%; get 50%)

⇒ greedy fractional soln has value 4

Quiz

Question: Let F = value of greedy fractional solution.
OPT = value of optimal (non-fractional) solution.

Which of the following is true?

(A) F = OPT for every knapsack instance

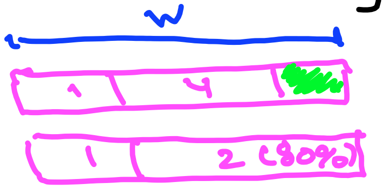
(B) F < OPT for every knapsack instance

(C) F ≤ OPT for every instance, and can be strict

(D) F ≥ OPT for every instance, and can be strict

Proof Sketch

Claim: greedy fractional solution at least as good as every non-fractional feasible solution.



(1) Let S = an arbitrary feasible solution.

(2) Suppose l units of knapsack filled by S with items not packed by the greedy fractional solution.

(3) Must be at least l units of knapsack filled by greedy fractional solution not packed by S

(4) By greedy criterion, items in (3) have larger bang-per-buck v_i/w_i than those in (2) [i.e., more valuable use of space]

(5) total value of greedy fractional solution at least that of S

Tim Roughgarden

Analysis of Greedy Heuristic

In Step 2, suppose our greedy algorithm picks the 1st k items (sorted by v_i/w_i).

Value of 3-step greedy algorithm $\geq$ total value of 1st k items

also is $\geq$ value of $(k+1)^{th}$ item (by step 3)

$\Rightarrow$ 2 × (value of 3-step greedy) $\geq$ total value of 1st (k+1) items

$\geq$ total value of greedy fractional soln

$\geq$ optimal knapsack solution

QED!

Analysis Is Tight

Example: $W = 1000$

$v_1 = 502$ $\quad v_2 = v_3 = 500$

$w_1 = 501$ $\quad w_2 = w_3 = 500$

$\Rightarrow$ 3-step greedy solution has value 502

$\Rightarrow$ optimal solution has value 1000

A Refined Analysis

Suppose: every item i has size $w_i \leq 10\% \cdot$ knapsack capacity W.

Consequence: if greedy algorithm fails to pack all items in Step 2, then the knapsack is $\geq 90\%$ full.

$\Rightarrow$ value of 2-step greedy algorithm $\geq 90\% \cdot$ value of greedy fractional soln

$\geq 90\% \cdot$ value of an optimal solution

[in general, if $\max_i w_i \leq \delta W$, then 2-step greedy value is $\geq (1-\delta) \cdot$ optimal]