All-Pairs Shortest Paths (APSP)

Algorithms: Design and Analysis, Part II

The Floyd-Warshall Algorithm

Quiz

Setup: Let A = 3-D array (indexed by i, j, k).

Intent: $A[i,j,k]$ = length of a shortest i-j path with all internal nodes in $\{1, 2, \ldots, k\}$. — or $+\infty$ if no such paths

Question: What is $A[i,j,0]$ if:

(1) $i = j$ (2) $(i,j) \in E$ (3) $i \neq j$ and $(i,j) \notin E$

(A) 0, 0, and $+\infty$

(B) 0, c_{ij}, and c_{ij}

(C) 0, c_{ij}, and $+\infty$

(D) $+\infty$, c_{ij}, and $+\infty$

Tim Roughgarden

The Floyd-Warshall Algorithm

Let A = 3-D array (indexed by i, j, k).

Base Cases: for all $i, j \in V$:

$$A[i,j,0] = \begin{cases} 0 & \text{if } i=j \\ c_{ij} & \text{if } (i,j) \in E \\ +\infty & \text{if } i \neq j \text{ and } (i,j) \notin E \end{cases}$$

For $k = 1$ to n
  For $i = 1$ to n
    For $j = 1$ to n

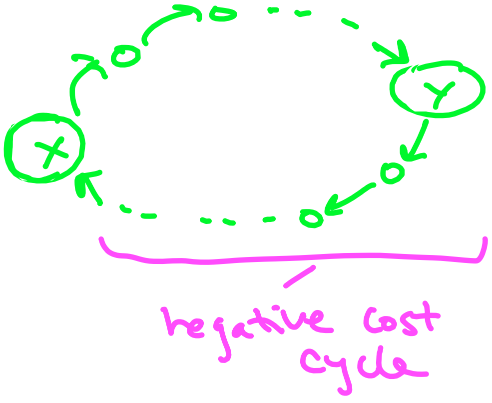
$$A[i,j,k] = \min \begin{cases} A[i,j,k-1] & \text{Case 1} \\ A[i,k,k-1] + A[k,j,k-1] & \text{Case 2} \end{cases}$$

Correctness: from optimal substructure + induction, as usual.

Running Time: $O(1)$ per subproblem, $O(n^3)$ overall.

Tim Roughgarden

Odds and Ends



Question #1: What if input graph G has a negative cycle?

Answer: will have $A[i,i,n] < 0$ for at least one $i \in V$ at end of algorithm.

Question #2: how to reconstruct a shortest i-j path?

Answer: in addition to A, have Floyd-Warshall compute $B[i,j]$ = max label of an internal node on a shortest i-j path for all $i,j \in V$.

[reset $B[i,j] = k$ if 2nd case of recurrence used to compute $A[i,j,k]$]

$\Rightarrow$ can use the $B[i,j]$'s to recursively reconstruct shortest paths!