The Bellman-Ford Algorithm

Optimal Substructure

Algorithms: Design and Analysis, Part II

Single-Source Shortest Path Problem, Revisited

Input: directed graph $G=(V,E)$, edge costs c_e [possibly negative], source vertex $s \in V$.

Goal: either

(A) for all destinations $v \in V$, compute the length of a shortest s-v path

focus of this + next video

OR

(B) output a negative cycle (excuse for failing to compute shortest paths)

later

Optimal Substructure (Informal)

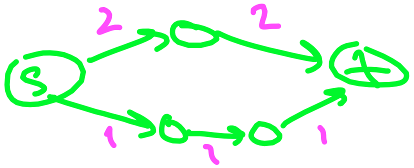
Intuition: exploit sequential nature of paths. Subpath of a shortest path should itself be shortest.

Issue: not clear how to define "smaller" & "larger" subproblems

Key idea: artificially restrict the number of edges in a path.

subproblem size $\longleftrightarrow$ number of permitted edges

Example



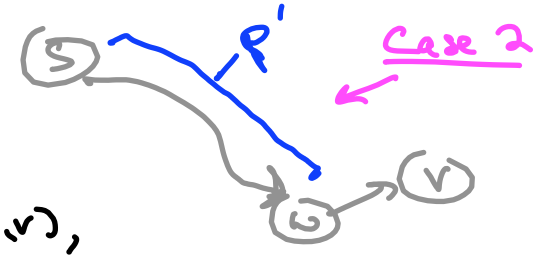
Optimal Substructure (Formal)

Lemma: Let $G = (V, E)$ be a directed graph with edge lengths c_e and source vertex s. {G might or might not have a negative cycle}

For every $v \in V$, $i \in \{1, 2, 3, \ldots\}$, let

P = shortest s-v path with at most i edges (cycles are permitted)

Case 1: if P has $\leq (i-1)$ edges, it is a shortest s-v path with $\leq (i-1)$ edges.



Case 2: if P has i edges with last hop (w, v), then P' is a shortest s-w path with $\leq (i-1)$ edges.

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Proof of Optimal Substructure

Case 1: by (obvious) contradiction.

Case 2: if $\underbrace{Q}_{\text{from } s \text{ to } w,\ \le (i-1) \text{ edges}}$ is shorter than P'

then $\underbrace{Q + (w,v)}_{\text{from } s \text{ to } v,\ \le i \text{ edges}}$ is shorter than $\underbrace{P' + (w,v)}_{= P}$

which contradicts the optimality of P.

QED!

Quiz

Question: how many candidates are there for an optimal solution to a subproblem involving the destination v?

A) 2

B) 1+ in-degree(v)

C) n - 1

D) n

1 from Case 1 +
1 from Case 2 for each choice of the final hop (w,v)