Advanced Union-Find

The Ackermann Function

Algorithms: Design and Analysis, Part II

Tarjan's Bound

Theorem: [Tarjan '75] With Union by Rank and path compression, m Union + Find operations take $O(m\,\alpha(n))$ time, where $\alpha(n)$ is the inverse Ackermann function.

(will define in this video)

proof in next video

The Ackermann Function

Aside: many different definitions, all more or less equivalent.

Will define $A_k(r)$ for all integers $k \geq 0$ and $r \geq 1$. (recursively)

Base case: $A_0(r) = r+1$ for all $r \geq 1$.

In general: for $k, r \geq 1$:

$$A_k(r) = \text{apply } A_{k-1} \text{ } r \text{ times to } r$$

$$= \underbrace{(A_{k-1} \circ A_{k-1} \circ \cdots \circ A_{k-1})}_{r\text{-fold composition}}(r)$$

Quiz: A_1

Quiz: $A_1(r)$ corresponds to what function of r?

(A) successor ($r \mapsto r+1$)

(B) doubling ($r \mapsto 2r$)

(C) exponentiation ($r \mapsto 2^r$)

(D) tower function ($r \mapsto 2^{2^{\cdot^{\cdot^{2}}}}\}$ r times)

$A_1(r) = \underbrace{(A_0 \circ \cdots \circ A_0)}_{\text{r-fold composition, add 1 each time}}(r)$

$= 2r$

Quiz: A_2

Question: what function does $A_2(r)$ correspond to?

(A) $r \mapsto 4r$

(B) $r \mapsto 2^r$

(C) $r \mapsto r2^r$

(D) $r \to 2^{2^{2^{\cdot^{\cdot^{2}}}}}$ } r times

$$A_2(r) = \underbrace{(A_1 \circ \cdots \circ A_1)}_{\text{r-fold composition, doubles each time}}(r)$$

$$= r2^r$$

Quiz: A_3

Question: what is $A_3(2)$?

recall $A_2(r) = r2^r$

(A) 8

(B) 1024

(C) 2048

(D) bigger than 2048

$$A_3(2) = A_2(A_2(2))$$
$$= A_2(8)$$
$$= 8 \cdot 2^8 = 2^{11} = 2048$$

In general: $A_3(r) = \underbrace{(A_2 \circ \cdots \circ A_2)}_{r \text{ times}}(r) \geq$ a tower of r 2's

$$\left.2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}\right\} r \text{ times}$$

A_4

$$A_4(2) = A_3(A_3(2))$$

$$= A_3(2048) \geq 2^{2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}} \Bigg\} \text{tower of height } 2048$$

<u>In general</u>: $A_4(r) = (\underbrace{A_3 \circ \cdots \circ A_3}_{r \text{ times}})(r)$

$\approx$ iterated tower function

(aka "wowzer" function)

The Inverse Ackermann Function

Definition: For every $n \geq 4$,

$\alpha(n)$ = minimum value of k such that $A_k(2) \geq n$.

$\alpha(n) = 1$ for $n = 4$ (since $A_1(2) = 4$)

$\alpha(n) = 2$ for $n = 5, 6, 7, 8$ (since $A_2(2) = 8$)

$\alpha(n) = 3$ for $n = 9, 10, 11, \ldots, 2048$ (since $A_3(2) = 2048$)

$\alpha(n) = 4$ n up to roughly a tower of 2's of height 2048

$\alpha(n) = 5$ for n up to ???

$\log^* n = 1$ for $n = 2$

$\log^* n = 2$ for $n = 3, 4$

$\log^* n = 3$ for $n = 5, \ldots, 16$

$\log^* n = 4$ for $n = 17, \ldots, 65536$

$\log^* n = 5$ for $n = 65537, \ldots, 2^{65536}$

$\vdots$

$\log^* n = 2048$ for these values of n

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