Algorithms: Design and Analysis, Part II

Minimum Spanning Trees

Correctness of Greedy Clustering

Correctness Claim

Theorem: single-link clustering finds the max-spacing k-clustering.

Proof: Let $C_1, \ldots, C_k$ = greedy clustering with spacing S.

Let $\hat{C}_1, \ldots, \hat{C}_k$ = arbitrary other clustering.

Need to show: spacing of $\hat{C}_1, \ldots, \hat{C}_k$ is $\leq S$.

Correctness Proof

Case 1: $\hat{C}_i$'s are the same as the C_i's [may be after renaming] $\Rightarrow$ has the same spacing S.

Case 2: otherwise, can find a point pair p,q such that

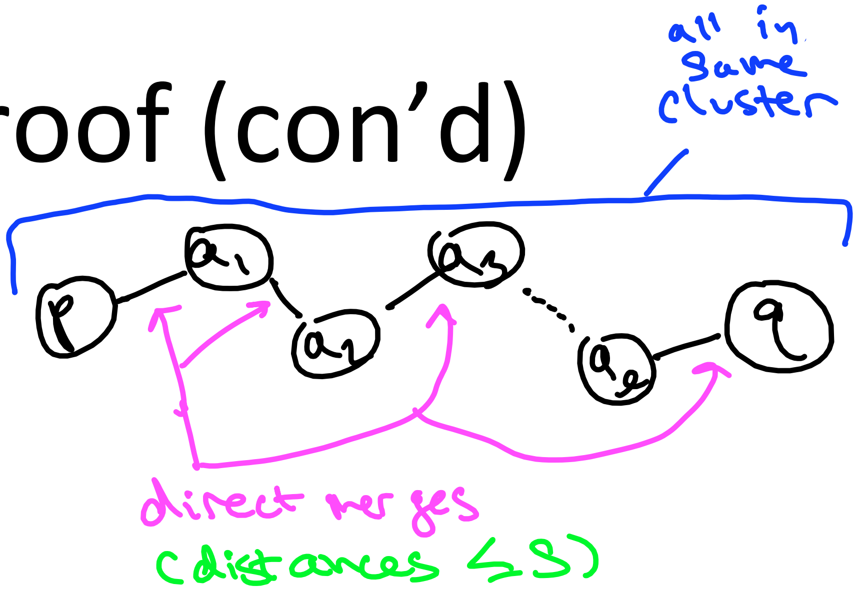
(A) p,q in the same greedy cluster C_i

(B) p,q in different clusters $\hat{C}_i, \hat{C}_j$

Property of greedy algorithm: if two points x,y "directly merged" at some point, then $d(x,y) \leq S$. [distance between merged point pairs only goes up]

Easy case: if p,q directly merged at some point, $S \geq d(p,q) \geq$ spacing of $\hat{C}_1, \dots, \hat{C}_k$

Correctness Proof (con'd)



Tricky case: p, q "indirectly merged" through multiple direct merges.

Let $p, a_1, \ldots, a_l, q$ be the path of direct greedy mergers connecting p & q.

Key point: Since $p \in \hat{C}_i$ and $q \notin \hat{C}_i$, $\exists$ consecutive pair a_j, a_{j+1} with $a_j \in \hat{C}_i$, $a_{j+1} \notin \hat{C}_i$

since a_j, a_{j+1} directly merged
since a_j, a_{j+1} separated

$\Rightarrow S \geq d(a_j, a_{j+1}) \geq$ spacing of $\hat{C}_1, \ldots, \hat{C}_k$

QED!